Characterizing the 4231-avoiding, shallow permutations

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Abstract: Depth measures the distance from a permutation to its symmetric group's identity and is known to be bound below by the average of two other measures of distance, length and reflection length. Permutations where this bound is an equality (i.e., permutations that are shallow) are known to not be characterizable by the containment and avoidance of patterns. However, using Hadjicostas and Monico's recursive definition for shallowness, we show that the subset of shallow permutations which avoid 4231 can be characterized in this way. Such permutations are the ones which avoid 4231, 3412, 34521, 54123, 365214, 541632, 7652143, and 5476321.

Background Information: Hadjicostas and Monico characterize shallowness as such: $\sigma \in S_{n>1}$ is shallow if it is constructed in one of the following ways:

1. Take a shallow permutation in S_{n-1} and add an n at the end
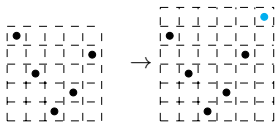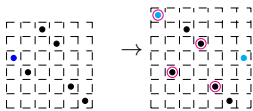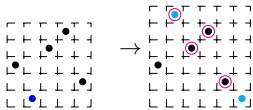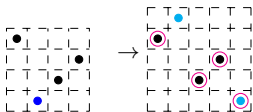

2. Take a shallow permutation in S_{n-1} and a left to right maximum or right to left minimum, and spawn two new "dots" above and to the right of it
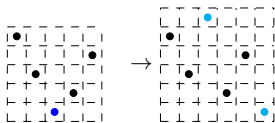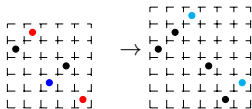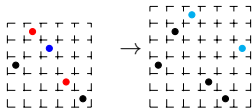
Method 1: Suppose that σ contains one of 4231, 3412, 34521, 54123, 365214, 541632, 7652143, and 5476321. Then we need to show one of the following:
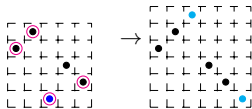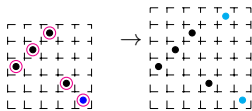
1. σ contains 4231:

2. σ is not shallow:

2a. σ was spawned from a dot that was neither left to right maximum nor right to left minimum:
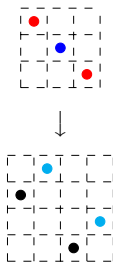
**2b.** The permutation σ spawned from contains another "bad" pattern:

Method 2: Suppose that σ is not shallow. Then we need to show that σ contains one of 4231, 3412, 34521, 54123, 365214, 541632, 7652143, and 5476321. The Hadjicostas and Monico characterization gives us two cases:

1. σ was spawned from a dot that was neither left to right maximum nor right to left minimum:

2. σ spawned from a left to right maximum or a right to left minimum in a permutation containing a bad pattern:
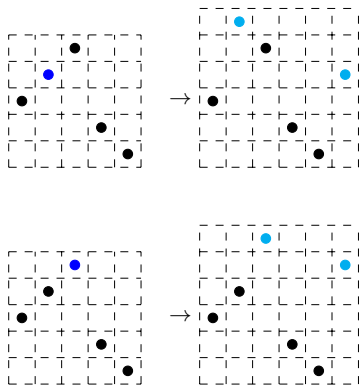
Every and every case yielded one of our desired conclusions.

Resulting Theorem: A 4231-avoiding permutation σ is shallow if and only if σ avoids 4231, 3412, 34521, 54123, 365214, 541632, 7652143, and 5476321.
i.e., 4231-avoiding permutations, which are shallow, form a pattern class.