Computing Extreme Values of Continuous & Discrete Fractional Gaussian Fields on Manifolds

Tyler Campos & Connor Marrs

Bowdoin College, Yale University, University of Connecticut

Abstract: Fractional Gaussian Fields (FGF's) are a family of stochastic processes that can be defined on any compact manifold M and model the behavior of random oscillations on M. Studying the extreme values of these fields is of great interest to a variety of subjects, and yet computing the distributions of the maxima and minima of the FGF are challenging, open problems. We define discrete fractional gaussian fields on manifolds M that converge to the continuous fractional field, and by numerically simulating the discrete fractional gaussian fields, we generate conjectures for the behavior of the extreme values of the continuous field.

Background Information:
For a sequence of real valued random variables, we have the following characterization

Theorem

A sequence $\{X_n\}$ of real valued random variables converges to a real valued random variable X if and only if for all bounded, continous $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$$

The Laplacian and the Fractional Gaussian Field

The Laplace-Beltrami operator on a compact manifold $\mathcal{M}$ is a self adjoint linear operator

$$\Delta_{\mathcal{M}} : C^{\infty}(\mathcal{M}) \to C^{\infty}(\mathcal{M})$$

that extends uniquely to a continuous operator on $L^2(\mathcal{M})$.
$\Delta_{\mathcal{M}}$ has a **discrete spectrum**:
Eigenfunctions $\{\phi_m\}_{m=1}^{\infty}$ of Δ form an orthonormal basis for $\mathcal{L}^2(\mathcal{M}, \mathcal{B}(\mathcal{M}), \mu)$:

$$f(x) = \sum_{m=1}^{\infty} \phi_m(x) \langle f, \phi_m \rangle_{L^2(\mathcal{M})}, \quad \langle \phi_m, \phi_n \rangle = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

with eigenvalues

$$0 = \lambda_0 > \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \to \infty$$

The **Fractional Gaussian Field with parameter** $s > d/4$ is a function

$$X_s(x) := \sum_{m=1}^{\infty} \frac{\phi_m(x) Z_m}{\lambda_m^s}$$

where $Z_m \sim \mathcal{N}(0, 1)$ are I.I.D. standard normal random variables.

Convergence of the discrete Fractional Gaussian Field

Let X_s be the fractional Gaussian field on the d dimensional torus $\mathbb{T}^d$. For a grid approximation of T^d, let Δ_n be the graph Laplacian. Then the discrete fractional Gaussian Field is defined analogously as

$$X_n^s(x) := \sum_{m=1}^{n} \frac{\vec{\phi}_m(x^-)Z_m}{\lambda_m^s}.$$

Theorem

For all $\vec{\theta}_1, \dots, \vec{\theta}_k \in \mathbb{T}^d$ *and* $s > \frac{d}{4}$, $\langle X_s^n(\vec{\theta}_1), \dots, X_s^n(\vec{\theta}_k)\rangle \xrightarrow{dist.} \langle X_s(\vec{\theta}_1), \dots, X_s(\vec{\theta}_k)\rangle$.

(a) X_{100}^{0}

(b) $X_{100}^{0.75}$

Convergence of the DFGF in the sense of Stochastic Processes

Definition

A sequence of random variable $\{X_n\}$ valued in $\mathscr{C}(S^1, \mathbb{R})$ converges to another random variable X in distribution (also valued in $\mathscr{C}(S^1, \mathbb{R})$) if and only if for each bounded and continuous function $f : \mathscr{C}(S^1, \mathbb{R}) \to \mathbb{R}$

$$\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$$

Theorem (Main Result)

Considering $\{X_s^n\}$ *and* X_s *as random variables valued in* $\mathscr{C}(S^1, \mathbb{R})$, $X_s^n \xrightarrow{dist.} X_s$.

- $\max_{t \in S^1} X_s^n \xrightarrow{dist.} \max_{t \in S^1} X_s$
- $\mathbb{E} \max_{t \in S^1} X_s^n \to \mathbb{E} \max_{t \in S^1} X_s$

Proof.

The map $\max : \mathscr{C}(S^1, \mathbb{R}) \to \mathbb{R}$ given by $f \mapsto \max_{t \in S^1} f(t)$ is continuous since if $\{f_n\}$ and f are continous functions on the circle and $f_n \to f$ uniformly, then $\max_{t \in S^1} f_n(t) \to \max_{t \in S^1} f(t)$.
(1) follows by the Continuous mapping theorem: if $\{X_n\}$ and X are random variables valued in a metric space (S, ρ), $X_n \xrightarrow{dist.} X$ and $\varphi : S \to \mathbb{R}$ is continous then $\varphi(X_n) \xrightarrow{dist.} \varphi(X)$.
(2) follows by the definition of convergence in distribution. $\square$