# Paths of the Fractional Gaussian Field on the Circle and Torus

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## Fractional Gaussian Fields (FGF)s

Let $(M, g)$ be a compact Riemannian manifold, and let $\mu$ be the uniform measure on $M$. If $-\Delta_M$ denotes the Laplace-Beltrami operator on $M$, then there exist eigenfunctions $\{\phi_i\}_{i=0}^{\infty}$ of $-\Delta_M$, with corresponding nonnegative eigenvalues $\{\lambda_i\}_{i=0}^{\infty}$, which form an orthonormal basis for the separable Hilbert space $L^2(M, \mathscr{B}(M), \mu)$. We may assume $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$.

Let $\mathcal{S}(M)$ denote the Schwartz class of smooth, compactly supported real-valued functions on $M$ whose Fourier coefficients decrease more rapidly than any polynomial, with its dual space $\mathcal{S}'(M)$. For any $s \geq 0$, we define the fractional Laplacian acting on functions $f \in \mathcal{S}(K)$ by

$$(-\Delta_M)^{-s} f(x) := \sum_{i=1}^{\infty} \frac{1}{\lambda_i^s} \phi_i(x) \int_M \phi_i(y) f(y) d\mu(y).$$

Let $\{W_i\}_{i=1}^{\infty}$ be an i.i.d sequence of standard normal Gaussian random variables on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Then

$$W(f) := \sum_{i=1}^{\infty} \int_M \phi_i(y) f(y) d\mu(y) W_i, \;\; f \in \mathcal{S}(M)$$

defines a **white noise** process on $L^2(M, \mathscr{B}(M), \mu)$. We define the **Fractional Gaussian Field** (FGF) with parameter $s \geq 0$ on $M$ acting on functions $f \in \mathcal{S}(M)$ by $X_s(f) = (-\Delta_M)^{-s} W = W((-\Delta_M)^{-s} f)$. Using self-adjointness of the operator $(-\Delta_M)^{-s}$, we compute

$$\begin{aligned} X_s(f) = W((-\Delta_M)^{-s} f) &= \sum_{i=1}^{\infty} \langle (-\Delta_M)^{-s} f, \phi_i \rangle_{L^2(M,\mu)} W_i \\ &= \sum_{i=1}^{\infty} \langle f, (-\Delta_M)^{-s} \phi_i \rangle_{L^2(M,\mu)} W_i = \sum_{i=1}^{\infty} \frac{\langle f, \phi_i \rangle_{L^2(M,\mu)}}{\lambda_i^s} W_i \end{aligned}$$

## Defining the FGF Pointwise

Thus far, the FGF has been defined as a random *distribution*. However, there are special cases in which one can define the FGF pointwise. Namely, when the parameter $s$ is greater than $\dim M/4$. Fix a point $p \in M$, and let $\delta_p$ denote the **Dirac measure** on $M$ concentrated at $p$. If $f \in \mathcal{S}(M)$, integration with respect to this measure gives

$$\int_M f d\delta_p = f(p).$$

Heuristically, we make the pointwise definition

$$\begin{aligned} X_s(p) &= X_s(\delta_p) \\ &= \sum_{i=1}^{\infty} \frac{\langle \delta_p, \phi_i \rangle_{L^2(M,\mu)}}{\lambda_i^s} W_i \\ &= \sum_{i=1}^{\infty} \frac{\phi_i(p)}{\lambda_i^s} W_i, \end{aligned}$$

and this series converges almost surely when $s > (\dim M)/4$. The goal of this poster presentation is to study of the path properties of the FGF in the case that it is defined pointwise.

## Paths of the FGF

Each point $\omega$ in our probability space $\Omega$ determines a function

$$M \to \mathbb{R} : p \mapsto X_s(p)(\omega).$$

A function of this form is called a **path** of the process $X_s$. Of course, these paths are defined only when $s > (\dim M)/4$. Numerical evidence suggests that the parameter $s$ controls not only when the FGF may be defined pointwise, but also the regularity of the sample paths. More specifically, greater values of the parameter $s$ imply greater regularity of almost every sample path of $X_s$. For instance, the figure below shows a path of $X_s$ on the circle $\mathbb{S}^1$ for three different values of the parameter $s$. Note that $\dim \mathbb{S}^1 = 1$, so in this case the critical value of the parameter is $1/4$.

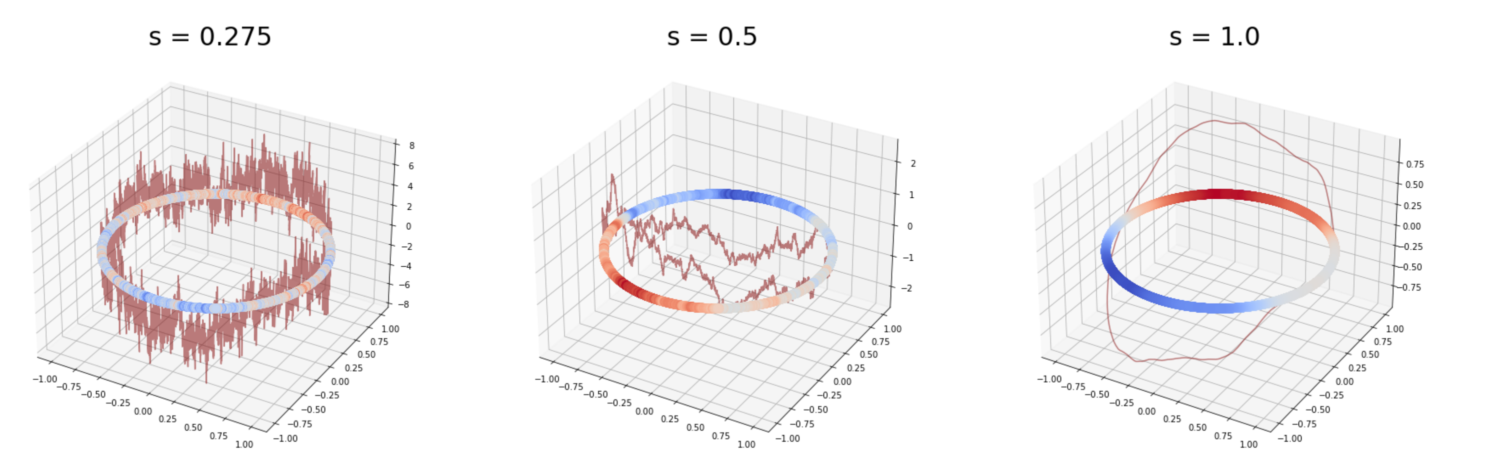


Fig. 1: A path of the FGF on the circle for the $s$-parameter values 0.275, 0.5, and 1.

As we see, the sample path depicted for the parameter value $0.275$ is highly irregular. For the parameter value of $0.5$, the sample path becomes a bit more regular, but when the parameter is as large as $1$ the sample path actually appears to be smooth. This suggests the fundamental question in our investigation: what are sufficient sizes of the parameter $s$ that will guarantee a particular degree of regularity in almost every sample path of $X_s$?

## Results for the Circle

The following are a collection of results we obtained regarding the regularity of paths of the FGF on the circle.

**Theorem.** *Let $s > 1/4$. Then there exist positive constants $\alpha, \beta$, and $\gamma$ such that*

$$\mathbb{E}\left[|X_s(\theta) - X_s(\phi)|^{\alpha}\right] \leq \beta \, |\theta - \phi|^{1+\gamma}$$

*for all $\theta, \phi \in \mathbb{S}^1$.*

By **Kolmogorov's Continuity Theorem**, this result implies that there exists a modification $\{\widetilde{X}_s(\theta)\}_{\theta \in \mathbb{S}^1}$ of $X_s$ whose paths are almost surely locally $\eta$-Hölder continuous for all $\eta \in (0, \beta/\alpha)$.

**Theorem.** *Let $\alpha \geq 1$ and suppose $s > \frac{1}{2} + \frac{1}{2\alpha}$. Then the paths of $X_s$ are almost surely $\alpha$-Hölder continuous functions. That is, for almost every $\omega \in \Omega$, there exists a constant $C_\omega \geq 0$ such that*

$$|X_s(\theta)(\omega) - X_s(\phi)(\omega)| \leq C_\omega \, |\theta - \phi|^{\alpha}$$

*for all $\theta, \phi \in \mathbb{S}^1$.*

A particular case of the theorem above is that when $s > 1$, the paths of $X_s$ on the circle are almost surely **Lipschitz** functions; i.e. they are Hölder continuous with exponent 1.

**Theorem.** *Let $k \in \mathbb{N}$, and suppose $s > \frac{k+1}{2}$. Then for almost every $\omega \in \Omega$, the path $X_s(\cdot)(\omega)$ is $k$ times differentiable, with derivative given by*

$$\sum_{n=1}^{\infty} \left( \frac{d^k}{d\theta^k}(\cos(n\theta)) \frac{Z_n(\omega)}{n^{2s}} + \frac{d^k}{d\theta^k}(\sin(n\theta)) \frac{Y_n(\omega)}{n^{2s}} \right).$$

*Here, $\{Z_n\}$ and $\{Y_n\}$ are sequences of IID standard Gaussian random variables. This is to say that the series defining the FGF may be differentiated term-by-term.*

## A Result for the Torus

Many of the results obtained for the FGF on the circle generalize with appropriate modifications to the $d$-dimensional torus $\mathbb{T}^d := \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{d\text{-times}}$. For instance, we have the following theorem:

**Theorem.** *If $X_s$ denotes the FGF on $\mathbb{T}^d$ and $s > \frac{d+1}{2}$, then the sample paths of $X_s$ are almost surely differentiable.*

In our simulations, we visualize the FGF on the torus $\mathbb{T}^2$ using a heat map.

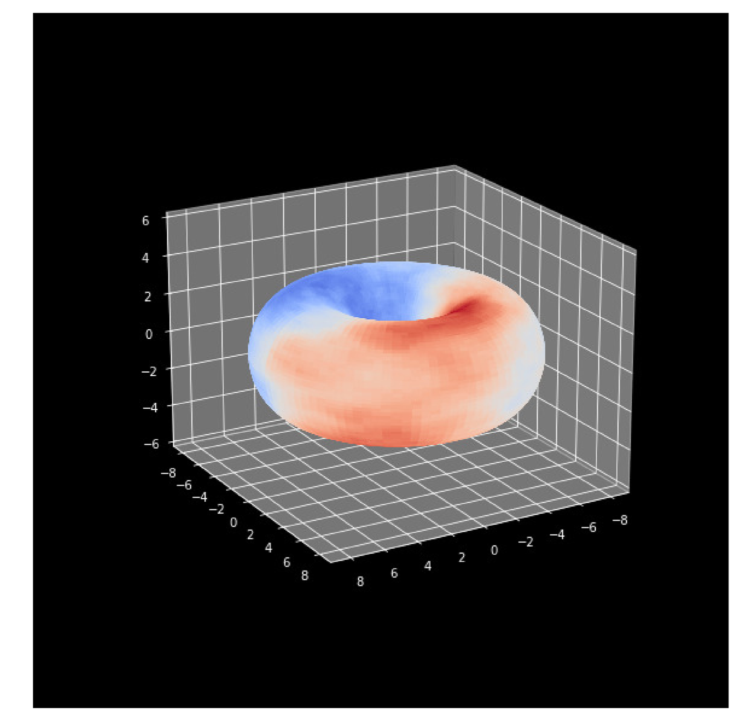

Fig. 2: A heat map depicting a path of the FGF on $\mathbb{T}^2$.

## Further Directions

- Regularity of paths of the FGF on the sphere $\mathbb{S}^2$.
- Paths of the FGF on arbitrary compact manifolds.
- Sobolev regularity of paths when $s \leq (\dim M)/4$.

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