Fractional Gaussian Fields (FGF)s

Let $(M, g)$ be a compact Riemannian manifold, and let $\mu$ be the uniform measure on $M$. If $-\Delta_M$ denotes the Laplace-Beltrami operator on $M$, then there exist eigenfunctions $\{\phi_i\}_i \subset (\Delta_M - 1)\mu$ with corresponding nonnegative eigenvalues $\lambda_i = \lambda_i(M)$, which form an orthonormal basis for the separable Hilbert space $L^2(M, \mu)$. We may assume $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$

Let $S(M)$ denote the Schwartz class of smooth, compactly supported real-valued functions on $M$ whose Fourier coefficients decrease more rapidly than any polynomial, with its dual space $S'(M)$. For any $s > 0$, we define the fractional Laplacian acting on functions $f \in S(M)$ by

$$(-\Delta_f)^s f(x) := \sum_{i=1}^{\infty} \lambda_i^s \phi_i(x) \langle f, \phi_i \rangle \mu(dx).$$

Let $\{W_i\}_i$ be an i.i.d sequence of standard normal Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$. Then

$$W(f) := \sum_{i=1}^{\infty} \langle f, \phi_i \rangle \phi_i W_i, \quad f \in S(M)$$

defines a white noise process on $L^2(M, \mu)$. We define the Fractional Gaussian Field (FGF) with parameter $s \geq 0$ on $M$ acting on functions $f \in S(M)$ by $X_s(f) = (-\Delta_f)^{-s/2} W = W(-\Delta_f)^{-s/2} f$. Using self-adjointness of the operator $(-\Delta_f)^{-s/2}$, we compute

$$X_s(f) = W(-\Delta_f)^{-s/2} f = \sum_{i=1}^{\infty} \langle (-\Delta_f)^{-s/2} f, \phi_i \rangle \phi_i W_i = \sum_{i=1}^{\infty} \langle f, \phi_i \rangle \phi_i (-\Delta_f)^{-s/2} W_i.$$

Defining the FGF Pointwise

Thus far, the FGF has been defined as a random distribution. However, there are special cases in which one can define the FGF pointwise. Namely, when the parameter $s$ is greater than $\dim M/4$. Fix a point $p \in M$, and let $\delta_p$ denote the Dirac measure on $M$ concentrated at $p$. If $f \in S(M)$, integration with respect to this measure gives

$$\int_M f d\delta_p = f(p).$$

Heuristically, we make the pointwise definition

$$X_s(p) = X_s(\delta_p) = \sum_{i=1}^{\infty} \langle \phi_i \rangle \phi_i (-\Delta_f)^{-s/2} W_i = \sum_{i=1}^{\infty} \frac{\phi_i(p)}{\phi_i(M)} W_i,$$

and this series converges almost surely when $s > (\dim M)/4$. The goal of this posterior presentation is to study of the path properties of the FGF in the case that it is defined pointwise.

Paths of the FGF

Each point $\omega$ in our probability space $\Omega$ determines a function $M \to \mathbb{R}: p \mapsto X_s(p)(\omega)$.

A function of this form is called a path of the process $X_s$. Of course, these paths are defined only when $s > (\dim M)/4$. Numerical evidence suggests that the parameter $s$ controls not only when the FGF may be defined pointwise, but also the regularity of the sample paths.

More specifically, greater values of the parameter $s$ imply greater regularity of almost every sample path of $X_s$. For instance, the figure below shows a path of $X_s$ on the circle $S^1$ for three different values of the parameter $s$. Note that $\dim S^1 = 1$, so in this case the critical value of the parameter is $1/4$.

As we see, the sample path depicted for the parameter value $s = 0.275$ is highly irregular. For the parameter value $s = 0.5$, the sample path becomes a bit more regular, but when the parameter is as large as $s = 1$ the sample path actually appears to be smooth. This suggests the fundamental question in our investigation: what are sufficient sizes of the parameter $s$ that will guarantee a particular degree of regularity in almost every sample path of $X_s$?

Results for the Circle

The following are a collection of results we obtained regarding the regularity of paths of the FGF on the circle.

**Theorem.** Let $s > 1/4$. Then there exist positive constants $\alpha$, $\beta$, and $\gamma$ such that

$$\mathbb{E} \left[ |X_s(\theta) - X_s(\phi)|^\beta \right] \leq \beta |\theta - \phi|^\alpha$$

for all $\theta, \phi \in S^1$.

By Kolmogorov’s Continuity Theorem, this result implies that there exists a modification $(X_s(\theta))_{\theta \in S^1}$ of $X_s$ whose paths are almost surely locally $\alpha$-Hölder continuous for all $\alpha \in (0, 1/2)$.

**Theorem.** Let $s \geq 1/4$ and suppose $s > 1/4$. Then the paths of $X_s$ are almost surely $\alpha$-Hölder continuous functions. That is, for almost every $\omega \in \Omega$, there exists a constant $C_{\alpha, \omega} \geq 0$ such that

$$|X_s(\theta)(\omega) - X_s(\phi)(\omega)| \leq C_{\alpha, \omega} |\theta - \phi|^\alpha$$

for all $\theta, \phi \in S^1$.

A particular case of the theorem above is that when $s > 1$, the paths of $X_s$ on the circle are almost surely Lipschitz functions; i.e., they are Hölder continuous with exponent 1.

**Theorem.** Let $k \in \mathbb{N}$, and suppose $s > k/2$. Then for almost every $\omega \in \Omega$, the path $X_s(\cdot)(\omega)$ is $k$ times differentiable, with derivative given by

$$\sum_{i=1}^{\infty} \frac{d^k}{dx^k} |\phi_i(n\theta)|^n \frac{Z_n(x)}{\lambda_i^n}.$$

Here, $\{Z_n\}$ and $\{Y_n\}$ are sequences of IID standard Gaussian random variables. This is to say that the series defining the FGF may be differentiated term-by-term.

A Result for the Torus

Many of the results obtained for the FGF on the circle generalize with appropriate modifications to the $d$-dimensional torus $T^d = \mathbb{R}^d / \mathbb{Z}^d$. For instance, we have the following theorem:

**Theorem.** If $X_s$ denotes the FGF on $T^d$ and $s > d/4$, then the sample paths of $X_s$ are almost surely differentiable.

In our simulations, we visualize the FGF on the torus $T^2$ using a heat map.

Further Directions

- Regularity of paths of the FGF on the sphere $S^2$.
- Paths of the FGF on arbitrary compact manifolds.
- Sobolev regularity of paths when $s \leq (\dim M)/4$.

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References


