Decategorification of $HFK_n(L)$
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Abstract: Using a definition of Euler characteristic for fractionally-graded complexes based on roots of unity, we show that the Euler characteristics of Dowlin’s “$sl(n)$-like” Heegaard Floer knot invariants $HFK_n$ recover both Alexander polynomial evaluations and $sl(n)$ polynomial evaluations at certain roots of unity for links in $S^3$. We show that the equality of these evaluations can be viewed as the decategorified content of the conjectured spectral sequences relating $sl(n)$ homology and $HFK_n$. This is joint work with Professor Andy Manion.

Background Information:
- $\overline{CFK}_n(L)$ is a single-graded $\mathbb{Q}[U_1, \ldots, U_{\ell-1}]$-complex constructed by Dowlin that conjecturally has spectral sequences from $sl_n$ homologies. $\overline{HFK}_n(L)$ is its homology.
- $\ell$ is the link’s number of components.
- $\overline{CFK}_n(L)$ is constructed from the double $\mathbb{Z}$-graded master complex $CFK_{UV}(L)$ of knot Floer homology with gradings $M$ and $A$, where
  - $M$ is the Maslov grading
  - $A$ is the Alexander grading
- $\overline{CFK}_n(L)$ has single $\mathbb{Z}$-grading $gr_n = -nM + 2(n-1)A$
Since the differential in $\overline{CFK}_n(L)$ has degree $n$, we divide the gradings $gr_n$ by $n$ so that Euler characteristics is preserved under homology.

**Definition**
For $\frac{1}{n}\mathbb{Z}$-graded vector space $C$, we define its Euler characteristic as

$$\chi(C) = \sum_{k \in \frac{1}{n}\mathbb{Z}} e^{\pi i k} \dim(C^k)$$

**Theorem (Dowlin)**
For knots $K$, $\overline{HFK}_n(K)$ is isomorphic to $gr_n$-graded knot Floer homology $\hat{HFK}(K)$

**Conjecture (Dowlin)**
For any link $L$ in $S^3$, there is a spectral sequence from $\overline{H}(L)$ to $\overline{HFK}(L)$ and from $H(L)$ to $HFK(L)$, where $\overline{H}(L)$ is the reduced HOMFLY-PT homology of $L$, $H(L)$ is the unreduced HOMFLY-PT homology of $L$, and $HFK(L)$ is the unreduced knot Floer homology of $K$. 
Main Results:
Theorem
For \( n \geq 2 \), the \( \frac{1}{n} \text{gr}_n \)-graded Euler characteristic of \( \text{HFK}_n(L) \) is
\[
e^{\pi i(1-\ell)/n} \Delta_L (e^{-2\pi i/n})
\]
where \( \Delta_L(t) \) is the Alexander polynomial and \( \ell \) is the number of components of \( L \).

Furthermore, we can interpret the conjectured spectral sequence from \( \text{H}_n(L) \) to \( \text{HFK}'_n(L) \) as a categorification of the equality \( \Delta_L(e^{\frac{2\pi i}{n}}) = P_{n,L}(e^{\frac{\pi i}{n}}) \)

Sketch of Proof:
Recall: \( \frac{\text{gr}_n}{n} = -M + (2 - \frac{2}{n})A \)
\[
\chi(\text{HFK}_n(K)) = \sum_{k \in \frac{1}{n} \mathbb{Z}} e^{\pi i k} \dim(\text{HFK}_n^k(K))
\]
\[
= \sum_{M,A \in \mathbb{Z}} (e^{\pi i})^{-M+(2-\frac{2}{n})A} \dim(\widehat{\text{HFK}}_{M,A}^k(K))
\]
\[
= \sum_{M,A \in \mathbb{Z}} (-1)^{M} q^A \dim(\widehat{\text{HFK}}_{M,A}^k(K)) \bigg|_{q=e^{-\frac{2\pi i}{n}}}
\]
\[
= \Delta_K(e^{-\frac{2\pi i}{n}}) = \Delta_K(e^{\frac{\pi i}{n}})
\]
The last equality follows from Proposition (Ozsváth & Szabó)

Proposition (Ozsváth & Szabó)
\[
\sum_{A,M} (-1)^M t^A \dim(\widehat{\text{HFK}}_{M,A}^k(L)) = (-1)^{\ell-1} t^{\frac{\ell-1}{2}} \left( 1 - t^{-1} \right)^{\ell-1} \Delta_L(t)
\]
Sketch of Proof:
Dowlin’s spectral sequence would give us $\chi(\overline{H}_n(L)) = \chi(\overline{HFK}'_n(L))$, hence $P_{n,L}(e^{\frac{\pi i}{n}}) = \Delta_L(e^{\frac{2\pi i}{n}})$

$$\chi(\overline{H}_n(L)) = \sum_{k \in \frac{1}{n}\mathbb{Z}} e^{\pi ik} \dim(\overline{H}_n^k(L))$$

$$= \sum_{\text{gr}_n, \text{gr}_v \in \mathbb{Z}} (e^{\pi i})^{\frac{\text{gr}_n}{n} + \text{gr}_v} \dim(\overline{H}_n^{\text{gr}_n, \text{gr}_v}(L))$$

$$= \sum_{\text{gr}_n, \text{gr}_v \in \mathbb{Z}} (-1)^{\text{gr}_v} e^{\frac{\pi i \text{gr}_n}{n}} \dim(\overline{H}_n^{\text{gr}_n, \text{gr}_v}(L))$$

$$= P_{n,L}(e^{\frac{\pi i}{n}})$$

Using proposition

Proposition (Khovanov & Rozansky)

$$\sum_{\text{gr}_n, \text{gr}_v} (-1)^{\text{gr}_v} q^{\text{gr}_n} \dim(\overline{H}_n^{\text{gr}_n, \text{gr}_v}(L)) = P_{n,L}(q)$$

where $P_{n,L}(q)$ is the $sl_n$ polynomial