Decategorification of $HFK_n(L)$

Larry Gu

University of Southern California

Abstract: Using a definition of Euler characteristic for fractionally-graded complexes based on roots of unity, we show that the Euler characteristics of Dowlin's "$sl(n)$-like" Heegaard Floer knot invariants HFK_n recover both Alexander polynomial evaluations and $sl(n)$ polynomial evaluations at certain roots of unity for links in S^3. We show that the equality of these evaluations can be viewed as the decategorified content of the conjectured spectral sequences relating $sl(n)$ homology and HFK_n. This is joint work with Professor Andy Manion.

Background Information:

- $\overline{CFK}_n(L)$ is a single-graded $\mathbb{Q}[U_1, ..., U_{\ell-1}]$-complex constructed by Dowlin that conjecturally has spectral sequences from sl_n homologies. $\overline{HFK}_n(L)$ is its homology.
- ℓ is the link's number of components.
- $\overline{CFK}_n(L)$ is constructed from the double $\mathbb{Z}$-graded master complex $CFK_{UV}(L)$ of knot Floer homology with gradings M and A, where
  - M is the Maslov grading
  - A is the Alexander grading
- $\overline{CFK}_n(L)$ has single $\mathbb{Z}$-grading $gr_n = -nM + 2(n-1)A$

Since the differential in $\overline{CFK}_n(L)$ has degree n, we divide the gradings gr_n by n so that Euler characteristics is preserved under homology.

Definition

For $\frac{1}{n}\mathbb{Z}$-graded vector space C, we define its Euler characteristic as

$$\chi(C) = \sum_{k \in \frac{1}{n}\mathbb{Z}} e^{\pi i k} \dim(C^k)$$

Theorem (Dowlin)

For knots K, $\overline{HFK}_n(K)$ is isomorphic to gr_n-graded knot Floer homology $\widehat{HFK}(K)$

Conjecture (Dowlin)

For any link L in S^3, there is a spectral sequence from $\overline{H}(L)$ to $\overline{HFK}(L)$ and from $H(L)$ to $HFK(L)$, where $\overline{H}(L)$ is the reduced HOMFLY-PT homology of L, $H(L)$ is the unreduced HOMFLY-PT homology of L, and $HFK(L)$ is the unreduced knot Floer homology of K.

Main Results:

Theorem

For $n \geq 2$, the $\frac{1}{n}\,\mathrm{gr}_n$-graded Euler characteristic of $\overline{HFK}_n(L)$ is

$$e^{\pi i(1-\ell)/n}\Delta_L(e^{-2\pi i/n})$$

where $\Delta_L(t)$ is the Alexander polynomial and ℓ is the number of components of L.

- Furthermore, we can interpret the conjectured spectral sequence from $\overline{H}_n(L)$ to $\overline{HFK}'_n(L)$ as a categorification of the equality $\Delta_L(e^{\frac{2\pi i}{n}}) = P_{n,L}(e^{\frac{\pi i}{n}})$

Sketch of Proof:

Recall: $\frac{gr_n}{n} = -M + (2 - \frac{2}{n})A$

$$\begin{aligned}
\chi(\overline{HFK}_n(K)) &= \sum_{k \in \frac{1}{n}\mathbb{Z}} e^{\pi i k} \dim(\overline{HFK}^k_n(K)) \\
&= \sum_{M,A \in \mathbb{Z}} (e^{\pi i})^{-M+(2-\frac{2}{n})A} \dim(\widehat{HFK}^{M,A}(K)) \\
&= \sum_{M,A \in \mathbb{Z}} (-1)^M q^A \dim(\widehat{HFK}^{M,A}(K))\Big|_{q=e^{-\frac{2\pi i}{n}}} \\
&= \Delta_K(e^{-\frac{2\pi i}{n}}) = \Delta_K(e^{\frac{2\pi i}{n}})
\end{aligned}$$

The last equality follows from Proposition (Ozsváth & Szabó)

Proposition (Ozsváth & Szabó)

$$\sum_{A,M}(-1)^M t^A \dim(\widehat{HFK}^{M,A}(L)) = (-1)^{\ell-1} t^{\frac{\ell-1}{2}} (1-t^{-1})^{\ell-1}\Delta_L(t)$$

Sketch of Proof:
Dowlin's spectral sequence would give us $\chi(\overline{H}_n(L)) = \chi(\overline{HFK}'_n(L))$, hence $P_{n,L}(e^{\frac{\pi i}{n}}) = \Delta_L(e^{\frac{2\pi i}{n}})$

$$\begin{aligned}
\chi(\overline{H}_n(L)) &= \sum_{k \in \frac{1}{n}\mathbb{Z}} e^{\pi i k} \dim(\overline{H}_n^k(L)) \\
&= \sum_{\mathbf{gr_n}, gr_v \in \mathbb{Z}} (e^{\pi i})^{\frac{\mathbf{gr_n}}{n} + gr_v} \dim(\overline{H_n}^{\mathbf{gr_n}, gr_v}(L)) \\
&= \sum_{\mathbf{gr_n}, gr_v \in \mathbb{Z}} (-1)^{gr_v} e^{\frac{\pi i \mathbf{gr_n}}{n}} \dim(\overline{H_n}^{\mathbf{gr_n}, gr_v}(L)) \\
&= P_{n,L}(e^{\frac{\pi i}{n}})
\end{aligned}$$

Using proposition

Proposition (Khovanov & Rozansky)

$$\sum_{\mathbf{gr_n}, gr_v} (-1)^{gr_v} q^{\mathbf{gr_n}} \dim(\overline{H_n}^{\mathbf{gr_n}, gr_v}(L)) = P_{n,L}(q)$$

where $P_{n,L}(q)$ is the sl_n polynomial