Abstract: To analyze the abundance of multidimensional data, tensor-based frameworks have been developed. Traditional matrix-based frameworks extract the most relevant features of vectorized data using the matrix-SVD. However, we may lose crucial high-dimensional relationships in this process. To facilitate efficient multidimensional feature extraction, we propose a projection-based classification algorithm using the t-SVDM, a tensor-based extension of the matrix-SVD. We apply our algorithm to the StarPlus fMRI dataset.

Matrix vs Tensor

Matrix Method:
- Uses matrix Singular Value Decomposition (SVD)
- Widely used in image processing
- Cannot identify relationships in higher dimensions

Tensor Method:
- Better representation of high-dimensional structure
- Flexibility in choosing a transformation matrix

Background

- The mode - k product [5] refers to the multiplication of a matrix along the k-th dimension of the tensor.
- $\star_M$ product: Given tensors $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ and $\mathcal{B} \in \mathbb{R}^{n_2 \times \ell \times n_4}$, and an invertible $M \in \mathbb{R}^{n_3 \times n_2}$:
  $$\mathcal{C} = \mathcal{A} \star_M \mathcal{B} = (\mathcal{A} \Delta \mathcal{B}) \times_3 M^{-1}$$
- Below displays the t-SVDM of a 3-D tensor [4]

Classification via Local t-SVDM [7]

Preprocessing

1. Split training data $\mathcal{A}$ into c distinct classes:
   $$\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_c$$
2. For each class i, compute t-SVDM and store first k basis elements:
   $$\mathcal{U}_i = U_i, \mathcal{S}_i, \mathcal{V}_i^T$$
   $$\mathcal{U}_{i,k} = U_i(:, 1 : k, :)$$

Classifying a Test Image $\mathcal{T}$

3. Project $\mathcal{T}$ onto space spanned by each class basis:
   $$\mathcal{P}_i = \mathcal{U}_{i,k} \mathcal{S}_i \mathcal{V}_i^T$$ for $i = 1, \ldots, c$
4. Categorize $\mathcal{T}$ as the class whose projection was "closest" to the original image:
   $$i^* = \arg\min_{i=1,\ldots,c} \| \mathcal{T} - \mathcal{P}_i \|_F$$
5. To measure the performance of our algorithm,
   $$\text{Accuracy} = \frac{\# \text{ correctly classified images}}{\# \text{ images}}$$
Intuition - MNIST [6]

Below is an illustration of two classes $\mathcal{A}_0$ (representing digits consisting of 0) and $\mathcal{A}_1$ (representing digits consisting of 1) go through the classification algorithm via local tensor SVD. Bases $\mathcal{U}_0$ and $\mathcal{U}_1$ are generated by digits from class 0 and class 1, respectively. We project $\mathcal{T}$ onto the spaces spanned by $\mathcal{U}_0$ and $\mathcal{U}_1$ and obtain $\mathcal{P}_0$ and $\mathcal{P}_1$, respectively. Absolute difference images $|\mathcal{T} - \mathcal{P}_0|$ and $|\mathcal{T} - \mathcal{P}_1|$ are generated by the absolute pixel difference between $\mathcal{T}$ and $\mathcal{U}_0$, and $\mathcal{T}$ and $\mathcal{P}_1$.

$\mathcal{A}_0$  $\mathcal{A}_1$  $\mathcal{T}$

$\mathcal{U}_0$  $\mathcal{U}_1$  $\mathcal{P}_0$  $|\mathcal{T} - \mathcal{P}_0|$  $\mathcal{P}_1$  $|\mathcal{T} - \mathcal{P}_1|$

Observations:
- $\mathcal{P}_0$ has characteristics of both digit 0 and digit 1
- $\mathcal{P}_1$ retains the characteristics of digit 1 only
- $\|\mathcal{T} - \mathcal{P}_0\|_F \approx 1.46 > \|\mathcal{T} - \mathcal{P}_1\|_F \approx 0.61$
- $\mathcal{T}$ classified as 1

StarPlus fMRI Data [2]

The StarPlus fMRI data consists of six human subjects completing 80 trials, each corresponding to the distinct cognitive tasks of viewing either a picture or a sentence.

Dimensions of Data

**Tensor:**
$\mathbf{X} = (\text{trials, } x, y, z, \text{time}) = (480, 64, 64, 8, 16)$

**Matrix (vectorized):**
$\mathbf{X} = (x \cdot y \cdot z \cdot \text{time, trials}) = (524288, 480)$
Power of Tensor Representations

Choices of $M$

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Advantage</th>
<th>Orthogonal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Banded</td>
<td>Temporal Dimension</td>
<td>No</td>
</tr>
<tr>
<td>Data-Dependent</td>
<td>Capture Structure Unique to Data</td>
<td>Yes</td>
</tr>
<tr>
<td>DCT</td>
<td>Separate Images with Varying Importance</td>
<td>Yes</td>
</tr>
<tr>
<td>DFT</td>
<td>Decompose Signals &amp; Catch Spatial Shift</td>
<td>Yes</td>
</tr>
<tr>
<td>Haar</td>
<td>Capture Sudden Transitions of Signals</td>
<td>Yes</td>
</tr>
<tr>
<td>Identity</td>
<td>Benchmark</td>
<td>Yes</td>
</tr>
<tr>
<td>Random Orthogonal</td>
<td>Assume No Data Structure; Might Give</td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td>Little Advantage over its Counterparts</td>
<td></td>
</tr>
<tr>
<td>ROI-Dependent</td>
<td>Understand Different ROIs’ roles</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Number of basis elements: Choose the $k$ largest singular values and their corresponding tensors to be our basis element.

Test accuracy with respect to number of basis elements for various choices of $M$ - product

- Traditional matrix method overlooks the intrinsic characteristics of fMRI images as brain slices over time are very interconnected
- Tensor method outperforms matrix method in test accuracy with:
  - appropriate choice of transformation matrix
  - small number of basis elements
Impact of Brain Regions

3D fMRI scan at single time point with twenty-five labeled Regions of Interest (ROI)

We also experiment with an ROI-dependent transformation matrix calculated from the most prominent ROI's in each trial.

Observations are the following:
- Best ROI’s vary depending on the subject
- No specific regions consistently improve performance in all subjects
- Illustrates how humans complete these cognitive tasks differently, demonstrating the difficulty of creating a good universal basis

Conclusion
- Dimension Extension: The original t-SVDM framework is proposed for only three-dimensions. Our paper provides definitions for a p-dimensional tensor and illustrates the usability of this framework for a five-dimensional fMRI dataset.
- Transformation Choices: We select the t-SVDM not only for its ability to process high-dimensional data, but also for the flexibility that the framework introduces via the $\ast$-product, which enables one to strategically choose a mathematical transformation based on the nature of the data being analyzed.
- Algorithm Flexibility: The t-SVDM and our proposed classification procedure is a mathematically justified framework that can be applied to any labeled high-dimensional data. Thus, all of the methods described in our paper can easily be extended to other similar classification tasks with labeled data.

Future Work
- Store compressible basis
- Combine our method with other nonlinear classification methods, e.g. neural networks
- Find orientation-independent tensor approaches
- Explore applications that would potentially be useful for medical diagnostic classification tasks.

Acknowledgement
This work was supported by the US National Science Foundation award DMS 2051019 and was completed during the "Computational Mathematics for Data Science" REU program in Summer 2021. We would like to thank Dr. Newman and the rest of the faculty supervising this REU program for their guidance and input throughout this project.

References