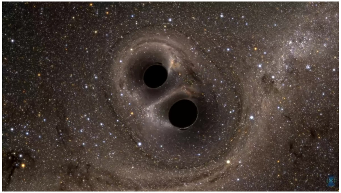
Viscous Relativistic Hydrodynamics

Jordan Marajh

Queen Mary, University of London

Abstract: Can the geometric ideas that have been successful in the study of relativistic perfect fluids be generalised to better understand viscosity at large scales? We aim to address this problem by considering how the Dynamical Systems behave; these arise from coupling Einstein's Field Equations to fluids with given equations of state motivated by recent progress in Astrophysics.



Background Information: Relativistic Hydrodynamics describes how fluids behave when near strong sources of gravity or when these fluids travel close to the speed of light. These settings are observed in physical phenomena such as black hole accretion disks or binary systems. When we fold viscosity of fluids into the problem, then the problems become much more difficult to understand.

The set-up

Einstein's Field Equations are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$$

where we consider the form of the stress-energy tensor to be

$$T^{E}_{\mu\nu} = (p+\rho) u_\mu u_\nu + p g_{\mu\nu}$$

in which we have p, ρ being the pressure and energy density of the fluid and u_μ as the four-velocity of the fluid in a co-moving frame. Here, the stress-energy tensor is one describing a perfect fluid. To add bulk viscosity into the picture, Eckart made a modification as follows to add in the bulk viscosity coefficient ζ

$$T^{Eck}_{\mu\nu} = (p+\rho) u_\mu u_\nu + p g_{\mu\nu} - \zeta (g_{\mu\nu} + u_\mu u_\nu) \nabla_\alpha u^\alpha.$$

Problem: This expression can yield superluminal signal propagation which is unphysical. In spite of the issues, this has been widely used in cosmology since its discovery.
Proposed Solution: Modifying this expression even further just slightly, as seen in work by Lichnerowicz and Disconzi, removes this problem by introducing a new quantity coined the dynamic velocity $C_\alpha = F u_\alpha$ where F is the index of the fluid and depends on the nature of the fluid itself.

Results:

- By choosing a spatially flat FLRW metric $ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$, thereby fixing the geometry on this hypersurface for time t, we can reduce Einstein's Field Equations $G_{\mu\nu} = T^{L-D}_{\mu\nu}$ to the Friedmann equation and evolution of density

$$\dot{H} = \frac{1}{2}(3\zeta FH + \zeta\dot{F} - (\rho + p))$$
$$\dot{\rho} = 3H(3\zeta FH + \zeta\dot{F} - (\rho + p))$$

where we define the Hubble parameter $H := \frac{\dot{a}}{a}$.
Now introducing the equation of state $p = \gamma\rho$ and $\zeta = \zeta_0\rho^\nu$, we see that the system of equations $(\dot{H}, \dot{\rho})$ admits an autonomous Dynamical System if F is also dependent on ρ.

- In order to introduce a shear viscosity term ϑ, we have to change our background metric from an isotropic to an anisotropic one and by supposing a new form of the stress-energy tensor, that is

$$T_{\mu\nu} = (p + \rho)u_\mu u_\nu + pg_{\mu\nu} - \left(\zeta - \frac{2}{3}\vartheta\right)\pi_{\mu\nu}\nabla_\alpha C^\alpha - \vartheta\pi^\alpha_\mu\pi^\beta_\nu(\nabla_\alpha C_\beta + \nabla_\beta C_\alpha).$$

Also, taking Bianchi I $ds^2 = -dt^2 + A_1^2(t)dx^2 + A_2^2(t)dy^2 + A_3^2(t)dz^2$, we get

$$\dot{H} = \frac{1}{2}(3\zeta FH + \zeta\dot{F} - (\rho + p)) + (\rho - 3H^2) - \frac{1}{3}\vartheta\dot{F}$$
$$\dot{\rho} = 3H(3\zeta FH + \zeta\dot{F} - (\rho + p)) - 4\vartheta F(\rho - 3H^2) - 2\vartheta\dot{F}$$

Comparing phase portraits and moving forwards

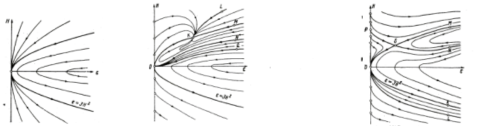
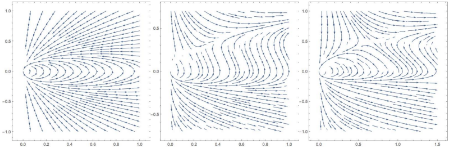


FIG. 1. Curves of ordinary Friedmann FIG. 2. Picture of the integral curves for the case $\nu < \frac{1}{2}$ FIG. 3. Integral curves for $\frac{1}{2} < \nu < 1$.



What's next? Studying the Dynamical Systems which arise from a fully-viscous stress-energy tensor

$$T^{ENS}_{\mu\nu} = (p+\rho)u_\mu u_\nu + p g_{\mu\nu} - \left(\zeta - \frac{2}{3}\vartheta\right)\pi_{\mu\nu}\nabla_\alpha C^\alpha - \vartheta \pi^\alpha_\mu \pi^\beta_\nu(\nabla_\alpha C_\beta + \nabla_\beta C_\alpha)$$
$$+ 2\vartheta\pi_{\mu\nu}u^\alpha\nabla_\alpha F - \kappa(q_\mu C_\nu + q_\nu C_\mu),$$

this may also even lead to a change of the background metric. Will the phase portraits look similar? What about the stability of the fixed points of the system? Do any regions of the phase portrait change drastically? Is the asymptotic behaviour the same for $\rho \to 0$ and $\rho \to \infty$?