Viscous Relativistic Fluids

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Abstract

The abstract serves both as a general introduction to the topic and as a brief, non-technical summary of the main results and their implications. Authors are advised to check the author instructions for the journal they are submitting to for word limits and if structural elements like subheadings, citations, or equations are permitted.

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1 Conventions

Throughout we have the standard conventions, the dimension of the spacetime is 4, Greek indices run from 0 to 3, Latin indices run from 1 to 3 and we employ the Einstein summation convention with repeated indices on tensors. Dots above a symbol refer to time derivatives and the pressure and density of a fluid are given by p and ρ respectively. Furthermore, we assume that we are working in a co-moving frame such that the four-velocity of the fluid is given by $u^0 = 1, u^i = 0$ and that it's scalar product . It is also worth noting that having a diagonal metric ensures that the stress-energy tensor, Ricci curvature tensors and Einstein tensor are also diagonal. Working in units where $8\pi G = c = 1$. Finally, dot refers to a time derivative.

We consider models introducing a dynamic velocity from the onset, if u^α is the four-velocity of a fluid then $C^\alpha = Fu^\alpha$ is the dynamic velocity where

F is the index of a fluid. This is some function of the pressure and density introduced to solve the superliminal signal problem when considering only the four-velocity.

2 FLRW and Bulk Viscosity

2.1 Metric Tensor and non-zero Christoffel symbols

Homogeneous and isotropic FLRW metric for (k=0) flat spacetime

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \tag{1}$$

Define the Hubble parameter as

$$H \equiv \frac{\dot{a}}{a}. \tag{2}$$

Given the metric, the non-zero Christoffel symbols are

$$\Gamma^0{}_{11} = \Gamma^0{}_{22} = \Gamma^0{}_{33} = \dot{a}a, \tag{3}$$

$$\Gamma^1{}_{10} = \Gamma^2{}_{20} = \Gamma^3{}_{30} = \frac{\dot{a}}{a} \equiv H. \tag{4}$$

Note that

$$\begin{aligned}\nabla_\alpha u^\alpha &= \partial_\alpha u^\alpha + \Gamma^\alpha{}_{\alpha\beta} u^\beta \\ &= \Gamma^\alpha{}_{\alpha\beta} u^\beta \\ &= \Gamma^\alpha{}_{\alpha 0} u^0 \\ &= \Gamma^\alpha{}_{\alpha 0} \\ &= 3\frac{\dot{a}}{a} \\ &\equiv 3H,\end{aligned} \tag{5}$$

and that

$$\begin{aligned}\nabla_\alpha C^\alpha &= \nabla_\alpha (F u^\alpha) \\ &= u^\alpha \nabla_\alpha F + F \nabla_\alpha u^\alpha \\ &= u^0 \nabla_0 F + F(3H) \\ &= \dot{F} + 3FH.\end{aligned} \tag{6}$$

2.2 Ricci Tensor and Scalar Curvature

The components of the Ricci tensor are

$$R_{00} = \partial_\alpha \Gamma^\alpha{}_{00} - \partial_0 \Gamma^\alpha{}_{0\alpha} + \Gamma^\alpha{}_{\alpha\beta}\Gamma^\beta{}_{00} - \Gamma^\alpha{}_{0\beta}\Gamma^\beta{}_{\alpha 0},$$

$$\begin{aligned}
&= -\partial_0 \Gamma^{\alpha}{}_{0\alpha} - \Gamma^{\alpha}{}_{0\beta}\Gamma^{\beta}{}_{\alpha 0}, \\
&= -\partial_0 \left(3\frac{\dot{a}}{a}\right) - 3\left(\frac{\dot{a}}{a}\right)^2, \\
&= -3\left(\frac{a\ddot{a} - \dot{a}^2}{a^2} + \frac{\dot{a}^2}{a^2}\right), \\
&= -3\frac{\ddot{a}}{a}, \qquad (7)
\end{aligned}$$

$$\begin{aligned}
R_{11} &= \partial_\alpha \Gamma^{\alpha}{}_{11} - \partial_1 \Gamma^{\alpha}{}_{1\alpha} + \Gamma^{\alpha}{}_{\alpha\beta}\Gamma^{\beta}{}_{11} - \Gamma^{\alpha}{}_{1\beta}\Gamma^{\beta}{}_{\alpha 1}, \\
&= \partial_0 \Gamma^{0}{}_{11} + \Gamma^{\alpha}{}_{\alpha 0}\Gamma^{0}{}_{11} - \left(\Gamma^{0}{}_{1\beta}\Gamma^{\beta}{}_{01} + \Gamma^{1}{}_{1\beta}\Gamma^{\beta}{}_{11}\right), \\
&= \partial_0(\dot{a}a) + \left(3\frac{\dot{a}}{a}\right)(\dot{a}a) - \left((\dot{a}a)\left(\frac{\dot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)(\dot{a}a)\right), \\
&= \dot{a}^2 + a\ddot{a} + 3\dot{a}^2 - 2\dot{a}^2, \\
&= a\ddot{a} + 2\dot{a}^2. \qquad (8)
\end{aligned}$$

By a very similar process, we have that

$$R_{22} = R_{33} = a\ddot{a} + 2\dot{a}^2, \qquad (9)$$

therefore, the Ricci scalar is

$$R = g^{\mu\nu}R_{\mu\nu} = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) \equiv 6\left(\frac{\ddot{a}}{a} + H^2\right). \qquad (10)$$

2.3 Stress-energy Tensor

The stress-energy tensor for describing bulk viscosity is given by

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu + p g_{\mu\nu} - \zeta(g_{\mu\nu} + u_\mu u_\nu)\nabla_\alpha C^\alpha, \qquad (11)$$

where ζ is the bulk viscosity coefficient.

Thus, the non-zero components are

$$T_{00} = \rho, \qquad (12)$$

$$T_{11} = T_{22} = T_{33} = a^2(p - \zeta\nabla_\alpha C^\alpha), \qquad (13)$$

and the trace is

$$T = g^{\mu\nu}T_{\mu\nu} = -\rho + 3(p - \zeta\nabla_\alpha C^\alpha) \qquad (14)$$

2.4 Friedmann Equations and Density Evolution Equation

The constraint equation (or second Friedmann equation) is obtained by looking at the (00)-Einstein Equation.

$$\begin{aligned} G_{00} = R_{00} - \frac{1}{2}g_{00}R = T_{00} \\ \iff -3\frac{\ddot{a}}{a} + \frac{1}{2}6\left(\frac{\ddot{a}}{a} + H^2\right) = \rho \\ \iff \rho = 3H^2 \end{aligned} \tag{15}$$

And the first Friedmann equation can be obtained by taking any of the identical (ii)-Einstein Equations, but it is first useful to know that

$$\begin{aligned} \dot{H} &= \partial_0\left(\frac{\dot{a}}{a}\right) \\ &= \frac{a\ddot{a} - \dot{a}^2}{a^2} \\ &= \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 \\ &= \frac{\ddot{a}}{a} - H^2 \\ \iff \frac{\ddot{a}}{a} &\equiv \dot{H} + H^2 \end{aligned} \tag{16}$$

So, using the above for ρ and the identity for $\dot{H}$.

$$\begin{aligned} G_{11} = R_{11} - \frac{1}{2}g_{11}R = T_{11} \iff \\ a\ddot{a} + 2\dot{a}^2 - \frac{1}{2}a^2\left(6\left(\frac{\ddot{a}}{a} + H^2\right)\right) = a^2(p - \zeta\nabla_\alpha C^\alpha) \\ \iff -2a\ddot{a} + 2\dot{a}^2 - a^2\rho = a^2(p - \zeta(\dot{F} + 3FH)) \\ \iff -2\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 - \rho = p - \zeta\dot{F} - 3\zeta FH \\ \iff -2(\dot{H} + H^2) + 2H^2 - \rho = p - \zeta\dot{F} - 3\zeta FH \\ \iff -2\dot{H} = \rho + p - \zeta\dot{F} - 3\zeta FH \\ \iff \dot{H} = -\frac{1}{2}\rho - \frac{1}{2}p + \frac{1}{2}\zeta\dot{F} + \frac{3}{2}\zeta FH \end{aligned} \tag{17}$$

For the density evolution equation, we look at the conservation law defined from the stress-energy tensor

$$\begin{aligned}
0 &= \nabla_\mu T^\mu{}_0 \\
&= \partial_\mu T^\mu{}_0 + \Gamma^\mu{}_{\alpha\mu} T^\alpha{}_0 - \Gamma^\beta{}_{0\mu} T^\mu{}_\beta \\
&= \partial_0 T^0{}_0 + \Gamma^\mu{}_{0\mu} T^0{}_0 - \left(\Gamma^1{}_{0\mu} T^\mu{}_1 + \Gamma^2{}_{0\mu} T^\mu{}_2 + \Gamma^3{}_{0\mu} T^\mu{}_3\right) \\
&= \partial_0(-\rho) + 3H(-\rho) - 3H(p - \zeta\nabla_\alpha C^\alpha) \\
&= -\dot{\rho} - 3H\rho - 3Hp + 3H\zeta(\dot{F} + 3FH) \\
&\iff \dot{\rho} = -3H\rho - 3Hp + 3\zeta\dot{F}H + 9\zeta FH^2 \qquad (18)
\end{aligned}$$

3 Bianchi I and Shear Viscosity

3.1 Metric and non-zero Christoffel symbols

Homogeneous and anisotropic Bianchi I metric for spacetime

$$ds^2 = -dt^2 + A_1^2(t)dx^2 + A_2^2(t)dy^2 + A_3^2(t)dz^2. \tag{19}$$

Given the metric, the non-zero Christoffel symbols are (for $i = 1, 2, 3$ ignoring the Einstein summation convention here for convenience)

$$\Gamma^0{}_{ii} = \dot{A}_i A_i, \tag{20}$$

$$\Gamma^i{}_{i0} = \frac{\dot{A}_i}{A_i}. \tag{21}$$

Define the Hubble parameter as

$$H \equiv \frac{1}{3} \sum_{i=1}^{3} \frac{\dot{A}_i}{A_i}. \tag{22}$$

Note that we have the following "invariants" from FLRW to Bianchi I,

$$\begin{aligned} \nabla_\alpha u^\alpha &= \Gamma^\alpha{}_{\alpha 0}, \\ &= \sum_{i=1}^{3} \frac{A_i}{A_i}, \\ &\equiv 3H, \end{aligned} \tag{23}$$

and that

$$\nabla_\alpha C^\alpha = \dot{F} + 3FH. \tag{24}$$

3.2 Ricci Tensor and Scalar Curvature

The components of the Ricci tensor are

$$\begin{aligned} R_{00} &= \partial_\alpha \Gamma^\alpha{}_{00} - \partial_0 \Gamma^\alpha{}_{0\alpha} + \Gamma^\alpha{}_{\alpha\beta}\Gamma^\beta{}_{00} - \Gamma^\alpha{}_{0\beta}\Gamma^\beta{}_{\alpha 0}, \\ &= -\partial_0 \Gamma^\alpha{}_{0\alpha} - \Gamma^\alpha{}_{0\beta}\Gamma^\beta{}_{\alpha 0}, \\ &= -\partial_0(3H) - \left(\Gamma^\alpha{}_{01}\Gamma^1{}_{\alpha 0} + \Gamma^\alpha{}_{02}\Gamma^2{}_{\alpha 0} + \Gamma^\alpha{}_{03}\Gamma^3{}_{\alpha 0}\right) \\ &= -\partial_0 \left(\sum_{i=1}^{3} \frac{\dot{A}_i}{A_i}\right) - \sum_{i=1}^{3} \left(\frac{\dot{A}_i}{A_i}\right)^2 \end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i=1}^{3} \partial_0 \left(\frac{\dot{A}_i}{A_i}\right) - \sum_{i=1}^{3}\left(\frac{\dot{A}_i}{A_i}\right)^2 \\
&= -\sum_{i=1}^{3} \left(\frac{A_i\ddot{A}_i - \dot{A}_i^{\,2}}{A_i^2}\right) - \sum_{i=1}^{3}\left(\frac{\dot{A}_i}{A_i}\right)^2 \\
&= \sum_{i=1}^{3} \left(-\frac{\ddot{A}_i}{A_i} + \left(\frac{\dot{A}_i}{A_i}\right)^2\right) - \sum_{i=1}^{3}\left(\frac{\dot{A}_i}{A_i}\right)^2 \\
&= -\sum_{i=1}^{3} \frac{\ddot{A}_i}{A_i}
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
R_{ii} &= \partial_\alpha \Gamma^\alpha{}_{ii} - \partial_i \Gamma^\alpha{}_{i\alpha} + \Gamma^\alpha{}_{\alpha\beta}\Gamma^\beta{}_{ii} - \Gamma^\alpha{}_{i\beta}\Gamma^\beta{}_{\alpha i}, \\
&= \partial_0 \Gamma^0{}_{ii} + \Gamma^\alpha{}_{\alpha 0}\Gamma^0{}_{ii} - \left(\Gamma^\alpha{}_{i0}\Gamma^0{}_{\alpha i} + \Gamma^\alpha{}_{i1}\Gamma^1{}_{\alpha i} + \Gamma^\alpha{}_{i2}\Gamma^2{}_{\alpha i} + \Gamma^\alpha{}_{i3}\Gamma^3{}_{\alpha i}\right) \\
&= \partial_0(\dot{A}_i A_i) + (3H)(\dot{A}_i A_i) - \left(\Gamma^i{}_{i0}\Gamma^0{}_{ii} + \Gamma^\alpha{}_{i1}\Gamma^1{}_{\alpha i} + \Gamma^\alpha{}_{i2}\Gamma^2{}_{\alpha i} + \Gamma^\alpha{}_{i3}\Gamma^3{}_{\alpha i}\right) \\
&= A_i\ddot{A}_i + \dot{A}_i^{\,2} + 3H\dot{A}_i A_i - \left(\dot{A}_i^{\,2} + \Gamma^\alpha{}_{i1}\Gamma^1{}_{\alpha i} + \Gamma^\alpha{}_{i2}\Gamma^2{}_{\alpha i} + \Gamma^\alpha{}_{i3}\Gamma^3{}_{\alpha i}\right) \\
&= A_i\ddot{A}_i + 3H\dot{A}_i A_i - \left(\Gamma^\alpha{}_{i1}\Gamma^1{}_{\alpha i} + \Gamma^\alpha{}_{i2}\Gamma^2{}_{\alpha i} + \Gamma^\alpha{}_{i3}\Gamma^3{}_{\alpha i}\right).
\end{aligned} \tag{26}
$$

Therefore, the remaining components are

$$
\begin{aligned}
R_{11} &= A_1\ddot{A}_1 + 3H\dot{A}_1 A_1 - \left(\Gamma^\alpha{}_{11}\Gamma^1{}_{\alpha 1} + \Gamma^\alpha{}_{12}\Gamma^2{}_{\alpha 1} + \Gamma^\alpha{}_{13}\Gamma^3{}_{\alpha 1}\right) \\
&= A_1\ddot{A}_1 + \left(\frac{\dot{A}_1}{A_1} + \frac{\dot{A}_2}{A_2} + \frac{\dot{A}_3}{A_3}\right)\dot{A}_1 A_1 - \dot{A}_1^{\,2} \\
&= A_1\left(\ddot{A}_1 + \dot{A}_1\left(\frac{\dot{A}_2}{A_2} + \frac{\dot{A}_3}{A_3}\right)\right).
\end{aligned} \tag{27}
$$

Similarly,

$$
R_{22} = A_2\left(\ddot{A}_2 + \dot{A}_2\left(\frac{\dot{A}_1}{A_1} + \frac{\dot{A}_3}{A_3}\right)\right), \tag{28}
$$

$$
R_{33} = A_3\left(\ddot{A}_3 + \dot{A}_3\left(\frac{\dot{A}_1}{A_1} + \frac{\dot{A}_2}{A_2}\right)\right). \tag{29}
$$

Finally, the Ricci scalar is given by

$$
R = g^{\mu\nu}R_{\mu\nu} = 2\left(\sum_{i=1}^{3}\frac{\ddot{A}_i}{A_i} + \frac{\dot{A}_1\dot{A}_2}{A_1A_2} + \frac{\dot{A}_2\dot{A}_3}{A_2A_3} + \frac{\dot{A}_1\dot{A}_3}{A_1A_3}\right) \tag{30}
$$

3.3 Stress-energy Tensor

The stress-energy tensor for describing shear viscosity is given by

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu + pg_{\mu\nu} - \left(\zeta - \frac{2}{3}\vartheta\right)\pi_{\mu\nu}\nabla_\alpha C^\alpha - \vartheta\pi^\beta{}_\mu\pi^\gamma{}_\nu(\nabla_\beta C_\gamma + \nabla_\gamma C_\beta), \tag{31}$$

where ζ is the bulk viscosity coefficient, ϑ is the shear viscosity coefficient and $\pi_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the projection.

It is useful to know that

$$\pi^\alpha{}_\nu = g^{\alpha\mu}\pi_{\mu\nu} = \delta^\alpha{}_\nu + u^\alpha u_\nu \tag{32}$$

so that in the case of $\nu = i = 1, 2, 3$, the projection with an index raised simplifies to the Kronecker delta as the spatial components of the four-velocity are zero ($u_i = 0$).

Moreover, we require

$$\begin{aligned}\nabla_\alpha C_\mu &= \partial_\alpha C_\mu - \Gamma^\beta{}_{\alpha\mu}C_\beta \\ &= \partial_\alpha(Fu_\mu) - \Gamma^0{}_{\alpha\mu}u_0 F \\ &= u_\mu\partial_\alpha F + \Gamma^0{}_{\alpha\mu}F\end{aligned} \tag{33}$$

Thus, the non-zero components are

$$\begin{aligned}T_{00} &= (p+\rho)u_0u_0 + pg_{00} - \left(\zeta - \frac{2}{3}\vartheta\right)(g_{00}+u_0u_0)\nabla_\alpha C^\alpha - \vartheta\pi^\beta{}_0\pi^\gamma{}_0(\nabla_\beta C_\gamma + \nabla_\gamma C_\beta), \\ &= \rho - \vartheta\delta^\beta{}_0\delta^\gamma{}_0(\nabla_\beta C_\gamma + \nabla_\gamma C_\beta), \\ &= \rho - \vartheta(\nabla_0 C_0 + \nabla_0 C_0), \\ &= \rho.\end{aligned} \tag{34}$$

And for $i = 1, 2, 3$

$$\begin{aligned}T_{ii} &= (p+\rho)u_iu_i + pg_{ii} - \left(\zeta - \frac{2}{3}\vartheta\right)(g_{ii}+u_iu_i)\nabla_\alpha C^\alpha - \vartheta\pi^\beta{}_i\pi^\gamma{}_i(\nabla_\beta C_\gamma + \nabla_\gamma C_\beta), \\ &= pA_i^2 - \left(\zeta - \frac{2}{3}\vartheta\right)A_i^2\nabla_\alpha C^\alpha - \vartheta\delta^\beta{}_i\delta^\gamma{}_i(\nabla_\beta C_\gamma + \nabla_\gamma C_\beta), \\ &= A_i^2 p' - \vartheta(\nabla_i C_i + \nabla_i C_i), \\ &= A_i^2 p' - 2\vartheta(u_i\partial_i F + \Gamma^0{}_{ii}F), \\ &= A_i^2 p' - 2\vartheta F\dot{A}_i A_i,\end{aligned} \tag{35}$$

where we have $p' = p - \left(\zeta - \frac{2}{3}\vartheta\right) \nabla_\alpha C^\alpha$ as the effective pressure. Raising an index yields (again ignoring the summation convention for convenience)

$$T^i_{\ i} = p' - 2\vartheta F \frac{\dot{A}_i}{A_i}. \tag{36}$$

The trace is then given by

$$T = g^{\mu\nu} T_{\mu\nu} = -\rho + 3p' - 2\vartheta F \sum_{i=1}^{3} \frac{\dot{A}_i}{A_i} \equiv -\rho + 3p' - 6\vartheta F H. \tag{37}$$

3.4 Friedmann Equations and Density Evolution Equation

The constraint equation (or second Friedmann equation) is obtained by looking at the (00)-Einstein Equation.

$$\begin{aligned} G_{00} &= R_{00} - \frac{1}{2} g_{00} R = T_{00} \\ \iff & -\sum_{i=1}^{3} \frac{\ddot{A}_i}{A_i} + \frac{1}{2} 2 \left(\sum_{i=1}^{3} \frac{\ddot{A}_i}{A_i} + \frac{\dot{A}_1 \dot{A}_2}{A_1 A_2} + \frac{\dot{A}_2 \dot{A}_3}{A_2 A_3} + \frac{\dot{A}_1 \dot{A}_3}{A_1 A_3} \right) = \rho \\ \iff & \rho = \frac{\dot{A}_1 \dot{A}_2}{A_1 A_2} + \frac{\dot{A}_2 \dot{A}_3}{A_2 A_3} + \frac{\dot{A}_1 \dot{A}_3}{A_1 A_3} \end{aligned} \tag{38}$$

And the first Friedmann equation can be obtained by taking the remaining (ii)–Einstein Equations and summing them. However, it is first useful to know that

$$\begin{aligned} p' &= p - \left(\zeta - \frac{2}{3}\vartheta\right) \nabla_\alpha C^\alpha, \\ &= p - \left(\zeta - \frac{2}{3}\vartheta\right) \left(\dot{F} + 3FH\right), \\ &= p - \zeta \dot{F} - 3F\zeta H + \frac{2}{3}\vartheta \dot{F} + 2\vartheta F H. \end{aligned} \tag{39}$$

As well as the following using 38

$$\begin{aligned} H^2 &= \frac{1}{9} \left(\sum_{i=1}^{3} \frac{\dot{A}_i}{A_i} \right)^2 \\ &= \frac{1}{9} \left(\sum_{i=1}^{3} \left(\frac{\dot{A}_i}{A_i} \right)^2 + 2 \left(\frac{\dot{A}_1 \dot{A}_2}{A_1 A_2} + \frac{\dot{A}_2 \dot{A}_3}{A_2 A_3} + \frac{\dot{A}_1 \dot{A}_3}{A_1 A_3} \right) \right) \end{aligned}$$

$$= \frac{1}{9}\left(\sum_{i=1}^{3}\left(\frac{\dot{A}_i}{A_i}\right)^2 + 2\rho\right)$$
$$\iff \sum_{i=1}^{3}\left(\frac{\dot{A}_i}{A_i}\right)^2 = 9H^2 - 2\rho \tag{40}$$

and the following using 40

$$\dot{H} = \frac{1}{3}\sum_{i=1}^{3}\partial_0\left(\frac{\dot{A}_i}{A_i}\right)$$
$$= \frac{1}{3}\sum_{i=1}^{3}\left(\frac{\ddot{A}_i}{A_i} - \left(\frac{\dot{A}_i}{A_i}\right)^2\right)$$
$$\iff 3\dot{H} = \sum_{i=1}^{3}\left(\frac{\ddot{A}_i}{A_i} - \left(\frac{\dot{A}_i}{A_i}\right)^2\right)$$
$$\iff \sum_{i=1}^{3}\frac{\ddot{A}_i}{A_i} = 3\dot{H} + \sum_{i=1}^{3}\left(\frac{\dot{A}_i}{A_i}\right)^2$$
$$\iff \sum_{i=1}^{3}\frac{\ddot{A}_i}{A_i} = 3\dot{H} + 9H^2 - 2\rho. \tag{41}$$

So, we have

$$G_{11} = R_{11} - \frac{1}{2}g_{11}R = T_{11} \iff$$
$$A_1\left(\ddot{A}_1 + \dot{A}_1\left(\frac{\dot{A}_2}{A_2} + \frac{\dot{A}_3}{A_3}\right)\right) - 2\frac{1}{2}A_1^2\left(\sum_{i=1}^{3}\frac{\ddot{A}_i}{A_i} + \frac{\dot{A}_1\dot{A}_2}{A_1A_2} + \frac{\dot{A}_2\dot{A}_3}{A_2A_3} + \frac{\dot{A}_1\dot{A}_3}{A_1A_3}\right)$$
$$= A_1^2p' - 2\vartheta F\dot{A}_1A_1$$
$$\iff -A_1^2\left(\frac{\ddot{A}_2}{A_2} + \frac{\ddot{A}_3}{A_3} + \frac{\dot{A}_2\dot{A}_3}{A_2A_3}\right) = A_1^2\left(p' - 2\vartheta F\frac{\dot{A}_1}{A_1}\right)$$
$$\iff \frac{\ddot{A}_2}{A_2} + \frac{\ddot{A}_3}{A_3} + \frac{\dot{A}_2\dot{A}_3}{A_2A_3} = -p' + 2\vartheta F\frac{\dot{A}_1}{A_1}. \tag{42}$$

Repeating with the (22) and (33) equations yields

$$\frac{\ddot{A}_1}{A_1} + \frac{\ddot{A}_3}{A_3} + \frac{\dot{A}_1\dot{A}_3}{A_1A_3} = -p' + 2\vartheta F\frac{\dot{A}_2}{A_2} \tag{43}$$
$$\frac{\ddot{A}_1}{A_1} + \frac{\ddot{A}_2}{A_2} + \frac{\dot{A}_1\dot{A}_2}{A_1A_2} = -p' + 2\vartheta F\frac{\dot{A}_3}{A_3} \tag{44}$$

respectively.

Summing and using 38, 39 and 40, we get

$$
\begin{aligned}
&2\sum_{i=1}^{3}\left(\frac{\ddot{A}_i}{A_i}\right)+\rho=-3\left(p-\zeta\dot{F}-3F\zeta H+\frac{2}{3}\vartheta\dot{F}+2\vartheta FH\right)+2\vartheta F\sum_{i=1}^{3}\frac{\dot{A}_i}{A_i}\\
&\iff 2(3\dot{H}+9H^2-2\rho)+\rho=-3\left(p-\zeta\dot{F}-3F\zeta H+\frac{2}{3}\vartheta\dot{F}+2\vartheta FH\right)+2\vartheta F(3H)\\
&\iff 6\dot{H}=-18H^2+3\rho-3p+3\zeta\dot{F}+9F\zeta H-2\vartheta\dot{F}\\
&\iff \dot{H}=-3H^2+\frac{1}{2}\rho-\frac{1}{2}p+\frac{1}{2}\zeta\dot{F}+\frac{3}{2}F\zeta H-\frac{1}{3}\vartheta\dot{F}.
\end{aligned}
\tag{45}
$$

For the density evolution equation, we again look at the conservation law defined from the stress-energy tensor and use 39

$$
\begin{aligned}
0&=\nabla_\mu T^\mu{}_0\\
&=\partial_\mu T^\mu{}_0+\Gamma^\mu{}_{\alpha\mu}T^\alpha{}_0-\Gamma^\beta{}_{0\mu}T^\mu{}_\beta\\
&=\partial_0 T^0{}_0+\Gamma^\mu{}_{0\mu}T^0{}_0-\left(\Gamma^1{}_{0\mu}T^\mu{}_1+\Gamma^2{}_{0\mu}T^\mu{}_2+\Gamma^3{}_{0\mu}T^\mu{}_3\right)\\
&=\partial_0(-\rho)+3H(-\rho)\\
&-\left[\frac{\dot{A}_1}{A_1}\left(p'-2\vartheta F\frac{\dot{A}_1}{A_1}\right)+\frac{\dot{A}_2}{A_2}\left(p'-2\vartheta F\frac{\dot{A}_2}{A_2}\right)+\frac{\dot{A}_3}{A_3}\left(p'-2\vartheta F\frac{\dot{A}_3}{A_3}\right)\right]\\
&=-\dot{\rho}-3H\rho-p'\sum_{i=1}^{3}\frac{\dot{A}_i}{A_i}+2\vartheta F\sum_{i=1}^{3}\left(\frac{\dot{A}_i}{A_i}\right)^2\\
&\iff \dot{\rho}=-3H\rho-3Hp'+2\vartheta F(9H^2-2\rho)\\
&\iff \dot{\rho}=-3H\rho-3H(p-\zeta\dot{F}-3F\zeta H+\frac{2}{3}\vartheta\dot{F}+2\vartheta FH)+18\vartheta FH^2-4\vartheta F\rho\\
&\iff \dot{\rho}=-3H\rho-3Hp+3\zeta\dot{F}H+9\zeta FH^2-2\vartheta\dot{F}-6\vartheta FH^2+18\vartheta FH^2-4\vartheta F\rho\\
&\iff \dot{\rho}=-3H\rho-3Hp+3\zeta\dot{F}H+9\zeta FH^2-2\vartheta\dot{F}+12\vartheta FH^2-4\vartheta F\rho
\end{aligned}
\tag{46}
$$

3.5 Scale Factors

Now, if we let $\dot{\varphi}=2\vartheta F$ and take 42 minus 43, we get

$$
\begin{aligned}
&\frac{\ddot{A}_2}{A_2}-\frac{\ddot{A}_1}{A_1}+\frac{\dot{A}_3}{A_3}\left(\frac{\dot{A}_2}{A_2}-\frac{\dot{A}_1}{A_1}\right)=-\dot{\varphi}\left(\frac{\dot{A}_2}{A_2}-\frac{\dot{A}_1}{A_1}\right)\\
&\iff \ddot{A}_2A_1-\ddot{A}_1A_2=-\frac{\dot{A}_3}{A_3}(\dot{A}_2A_1-\dot{A}_1A_2)-\dot{\varphi}(\dot{A}_2A_1-\dot{A}_1A_2)\\
&\iff \frac{\ddot{A}_2A_1-\ddot{A}_1A_2}{\dot{A}_2A_1-\dot{A}_1A_2}=-\frac{\dot{A}_3}{A_3}-\dot{\varphi}
\end{aligned}
$$

$$\begin{aligned}
&\iff \frac{d}{dt}[\ln(\dot{A}_2A_1-\dot{A}_1A_2)] = -\frac{\dot{A}_3}{A_3}-\dot{\varphi}\\
&\iff \frac{d}{dt}[\ln(\dot{A}_2A_1-\dot{A}_1A_2)] - \left(\frac{\dot{A}_1}{A_1}+\frac{\dot{A}_2}{A_2}\right) = -\left(\frac{\dot{A}_1}{A_1}+\frac{\dot{A}_2}{A_2}+\frac{\dot{A}_3}{A_3}\right)-\dot{\varphi}\\
&\iff \frac{d}{dt}[\ln(\dot{A}_2A_1-\dot{A}_1A_2)] - \frac{d}{dt}[\ln A_1A_2] = -\frac{d}{dt}[\ln A_1A_2A_3]-\dot{\varphi} \qquad (47)
\end{aligned}$$

Integrating yields

$$\begin{aligned}
&\ln\left(\frac{\dot{A}_2A_1-\dot{A}_1A_2}{A_1A_2}\right) = -\ln(A_1A_2A_3)-\varphi+C_{12}\\
&\iff \frac{\dot{A}_2}{A_2}-\frac{\dot{A}_1}{A_1} = \frac{C'_{12}}{A_1A_2A_3}e^{-\varphi}, \qquad (48)
\end{aligned}$$

where $C'_{12} = e^{C_{12}}, C_{12} \in \mathbb{R}$.

Similarly, by considering 44 minus 43, one arrives at

$$\frac{\dot{A}_2}{A_2}-\frac{\dot{A}_3}{A_3} = \frac{C'_{23}}{A_1A_2A_3}e^{-\varphi}. \qquad (49)$$

Adding 48 and 49 gives

$$\begin{aligned}
&2\frac{\dot{A}_2}{A_2}-\left(\frac{\dot{A}_1}{A_1}+\frac{\dot{A}_3}{A_3}\right) = \frac{C'_{12}+C'_{23}}{A_1A_2A_3}e^{-\varphi}\\
&\iff 3\frac{\dot{A}_2}{A_2}-3H = \frac{C'_{12}+C'_{23}}{A_1A_2A_3}e^{-\varphi}\\
&\iff \frac{\dot{A}_2}{A_2} = H+\frac{S_2}{A_1A_2A_3}e^{-\varphi}. \qquad (50)
\end{aligned}$$

where $S_2 = \frac{1}{3}(C'_{12}+C'_{23})$. Of course, by beginning with 42 and 44 or 43 and 44, one would arrive at similar results, giving

$$\frac{\dot{A}_3}{A_3} = H+\frac{S_3}{A_1A_2A_3}e^{-\varphi} \qquad (51)$$

$$\frac{\dot{A}_1}{A_1} = H+\frac{S_1}{A_1A_2A_3}e^{-\varphi} \qquad (52)$$

with S_1 and S_3 defined like S_2.

Notice that $\sum_{i=1}^{3} S_i = 0$ is necessary by taking the sum of the above linear, first-order ODEs for the scale factors.