Lyapunov-Type Inequalities for a Third Order Nonlinear Equation

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Abstract: We derive Lyapunov-type inequalities for general third order nonlinear equations involving multiple ψ-Laplacian operators of the form

$$\begin{cases} -(\psi_2((\psi_1(u'))'))' = q(x)f(u) \\ u(a) = u(b) = (\psi_1(u'))'(\xi) = 0 \end{cases} \quad (1)$$

for $\xi \in [a, b]$. Additionally,

(H1) ψ_1 and ψ_2 are odd and increasing functions, $\psi_1(xy) \leq \psi_1(x)\psi_1(y)$, and $\frac{1}{\psi_1(x)}$ is a convex function.

(H2) $q(x)$ and $f(u)$ are continuous. Furthermore, $f(u)$ is odd and satisfies $uf(u) > 0$.

These new inequalities generalizes previously obtained results, and the proofs utilize a different technique than most other works in the literature. Furthermore, using the obtained inequalities, we obtain a lower bound for the number of zeroes and properties of oscillatory solutions.

Figure 1: Aleksandr M. Lyapunov (1857-1918), Russian mathematician and physicist.

Background

In 1892 A. Lyapunov developed the sufficient condition for the existence of a nontrivial solution for the following problem.

$$u'' + q(x)u = 0, \quad u(a) = u(b) = 0. \tag{2}$$

His result is referred to as the Lyapunov Inequality, and states that if (2) has a nontrivial solution, then

$$\int_a^b |q(x)|\,dx > \frac{4}{b-a}.$$

Various improvements and generalizations have been made to the inequality, as well as extensions to higher order cases. Many proofs utilize the Cauchy-Schwartz Inequality, Holder's Inequality, or some variation.

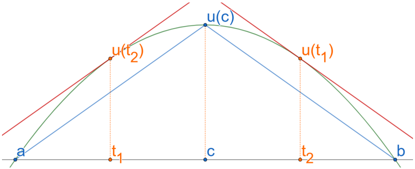

Figure 2: In the proof, we make a standard assumption that u is positive in the interval, and therefore attains a maximum. Moreover, we use the Mean Value Theorem and Jensen's Inequality in the first steps of our proof.

One significant generalization of the inequality was the improvement from $|q(x)|$ to $q_+(x) = \max\{0, q(x)\}$,

$$\int_a^b q_+(x)\,dx > \frac{4}{b-a}.$$

In the context of eigenvalue problems, these inequalities provide a lower bound for the eigenvalues.

Main Result

Theorem (Behrens and Dhar, 2021)

Assume (1) has a nontrivial solution u. Then

$$\int_a^{\xi} q_-(x)\Phi(u)\,dx + \int_{\xi}^{b} q_+(x)\Phi(u)\,dx > \psi_2\left(\frac{\frac{2}{b-a}}{\psi_1\left(\frac{b-a}{2}\right)}\right),$$

where $\Phi(u) = \frac{f(u)}{\psi_2(\psi_1(u))}$.

Furthermore, we have also proved the case when a nontrivial solution of (1) has three consecutive zeros. This is currently the most general result for a third order problem.

Observe that as $\xi \to a$,

$$\int_a^{b} q_+(x)\Phi(u) > \psi_2\left(\frac{\frac{2}{b-a}}{\psi_1\left(\frac{b-a}{2}\right)}\right).$$

Similarly as $\xi \to b$,

$$\int_a^{b} q_-(x)\Phi(u) > \psi_2\left(\frac{\frac{2}{b-a}}{\psi_1\left(\frac{b-a}{2}\right)}\right).$$

When ψ_2, ψ_1 and f are power functions satisfying $\Phi(u) = 1$, we have

$$\int_a^{\xi} q_-(x)\,dx + \int_{\xi}^{b} q_+(x)\,dx > \left(\frac{2}{b-a}\right)^{\alpha}.$$

Applications

Oscillatory Solutions

Let u be an oscillatory solution of (1). Let $\{t_k\}_{k=1}^{\infty}$ be an increasing sequence of zeros in $[0, \infty)$. Assume there exists a $\sigma \geq 1$ such that for any $M > 0$ we have

$$\int_t^{t+M} |q(s)|^\sigma \, ds \to 0 \text{ as } t \to \infty,$$

then $t_{n+2} - t_n \to \infty$ as $n \to \infty$.

Number of Zeros

Let u be a nontrivial solution of (1). Let $\{t_k\}_{k=1}^{2N+1}, N \geq 1$ be an increasing sequence of zeroes of u in a compact interval $[a, b]$. Also, let ψ_2 be convex. Then

$$\sum_{k=1}^{N} \max_{\xi_k \in [t_{2k-1}, t_{2k+1}]} \left\{ \int_{t_{2k-1}}^{\xi_k} q_-(x)\Phi(u)\, dx + \int_{\xi_k}^{t_{2k+1}} q_+(x)\Phi(u)\, dx \right\} > N\psi_2 \left(\frac{\frac{2N}{b-a}}{\psi_1\left(\frac{b-a}{2N}\right)} \right).$$

We also conclude that for the special case of the p-Laplacian, the location of the maximum of a nontrivial solution cannot be too close to the boundary.