Lyapunov-Type Inequalities for a Third Order Nonlinear Equation
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Abstract: We derive Lyapunov-type inequalities for general third order nonlinear equations involving multiple $\psi$-Laplacian operators of the form

$$\begin{cases}
-(\psi_2((\psi_1(u'))'))' = q(x)f(u) \\
u(a) = u(b) = (\psi_1(u'))'(\xi) = 0
\end{cases}$$

(1)

for $\xi \in [a, b]$. Additionally,

(H1) $\psi_1$ and $\psi_2$ are odd and increasing functions, $\psi_1(xy) \leq \psi_1(x)\psi_1(y)$, and $\frac{1}{\psi_1(x)}$ is a convex function.

(H2) $q(x)$ and $f(u)$ are continuous. Furthermore, $f(u)$ is odd and satisfies $uf(u) > 0$.

These new inequalities generalize previously obtained results, and the proofs utilize a different technique than most other works in the literature. Furthermore, using the obtained inequalities, we obtain a lower bound for the number of zeroes and properties of oscillatory solutions.

Figure 1: Aleksandr M. Lyapunov (1857-1918), Russian mathematician and physicist.
In 1892 A. Lyapunov developed the sufficient condition for the existence of a nontrivial solution for the following problem.

\[ u'' + q(x)u = 0, \quad u(a) = u(b) = 0. \]  \hspace{1cm} (2)

His result is referred to as the Lyapunov Inequality, and states that if (2) has a nontrivial solution, then

\[ \int_a^b |q(x)| \, dx > \frac{4}{b - a}. \]

Various improvements and generalizations have been made to the inequality, as well as extensions to higher order cases. Many proofs utilize the Cauchy-Schwartz Inequality, Holder’s Inequality, or some variation.

One significant generalization of the inequality was the improvement from \(|q(x)|\) to \(q_+(x) = \max\{0, q(x)\}\),

\[ \int_a^b q_+(x) \, dx > \frac{4}{b - a}. \]

In the context of eigenvalue problems, these inequalities provide a lower bound for the eigenvalues.
Main Result

**Theorem (Behrens and Dhar, 2021)**

Assume (1) has a nontrivial solution $u$. Then

$$
\int_a^\xi q_-(x)\Phi(u)\,dx + \int_{\xi}^b q_+(x)\Phi(u)\,dx > \psi_2 \left( \frac{2}{b-a} \right) \left( \frac{\psi_1}{\psi_1 (b-a)} \right),
$$

where $\Phi(u) = \frac{f(u)}{\psi_2(\psi_1(u))}$. Furthermore, we have also proved the case when a nontrivial solution of (1) has three consecutive zeros. This is currently the most general result for a third order problem.

Observe that as $\xi \to a$,

$$
\int_a^b q_+(x)\Phi(u) > \psi_2 \left( \frac{2}{b-a} \right) \left( \frac{\psi_1}{\psi_1 (b-a)} \right).$$

Similarly as $\xi \to b$,

$$
\int_a^b q_-(x)\Phi(u) > \psi_2 \left( \frac{2}{b-a} \right) \left( \frac{\psi_1}{\psi_1 (b-a)} \right).
$$

When $\psi_2, \psi_1$ and $f$ are power functions satisfying $\Phi(u) = 1$, we have

$$
\int_a^\xi q_-(x)\,dx + \int_{\xi}^b q_+(x)\,dx > \left( \frac{2}{b-a} \right)^\alpha.
$$
Applications

Oscillatory Solutions

Let \( u \) be an oscillatory solution of (1). Let \( \{t_k\}_{k=1}^{\infty} \) be an increasing sequence of zeros in \([0, \infty)\). Assume there exists a \( \sigma \geq 1 \) such that for any \( M > 0 \) we have

\[
\int_t^{t+M} |q(s)|^\sigma \, ds \to 0 \text{ as } t \to \infty,
\]
then \( t_{n+2} - t_n \to \infty \text{ as } n \to \infty. \)

Number of Zeros

Let \( u \) be a nontrivial solution of (1). Let \( \{t_k\}_{k=1}^{2N+1}, N \geq 1 \) be an increasing sequence of zeroes of \( u \) in a compact interval \([a, b]\). Also, let \( \psi_2 \) be convex. Then

\[
\sum_{k=1}^{N} \max_{\xi_k \in [t_{2k-1}, t_{2k+1}]} \left\{ \int_{t_{2k-1}}^{\xi_k} q_-(x)\Phi(u) \, dx + \int_{\xi_k}^{t_{2k+1}} q_+(x)\Phi(u) \, dx \right\} > N\psi_2 \left( \frac{2N}{\psi_1 \left( \frac{b-a}{2N} \right)} \right).
\]

We also conclude that for the special case of the \( p \)-Laplacian, the location of the maximum of a nontrivial solution cannot be too close to the boundary.