Properties of Reduced Convex Hulls

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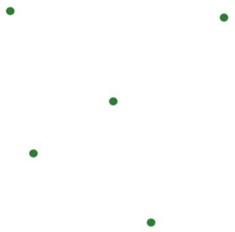
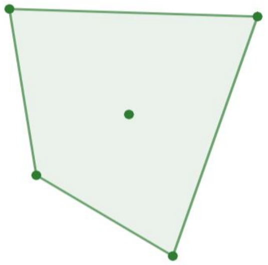
Abstract: We explore the properties of randomly-generated reduced convex hulls, both when the number of underlying points is fixed and in the asymptotic case. The results allow us to describe the number of vertices of a random reduced convex hull as either a function of the number of points or the hull parameter μ.

Background Information: Given a set of points $X = \{x_1, x_2, \ldots, x_N\}$ we define the *convex* hull of X to be the smallest convex set containing X, or equivalently,

$$\mathrm{CH}(X) = \left\{ \sum_{i=1}^{N} \alpha_i x_i : \alpha_i \geq 0, \sum_{i=1}^{N} \alpha_i = 1 \right\}.$$



$|X| = 5$ points

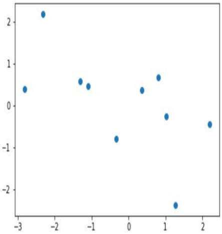


$\mathrm{CH}(X)$

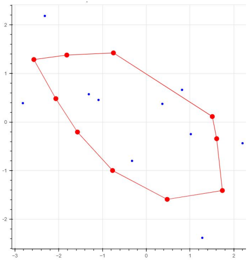
Sometimes we want to "shrink" the convex hull of a set of points in a non-uniform way. Given the same set of points X, we define the *reduced convex hull* of X to be

$$\mathrm{RCH}(X, \mu) = \left\{ \sum_{i=1}^{N} \alpha_i x_i : 0 \leq \alpha_i \leq \mu, \sum_{i=1}^{N} \alpha_i = 1 \right\}.$$

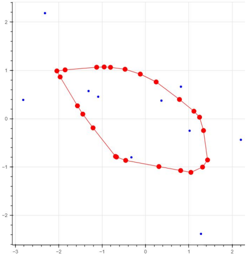
Notice when $\mu = 1$ we obtain the full convex hull of X. Reduced convex hulls originated in the study of binary classification problems for which the data sets overlap.



$|X| = 10$ points



$\mathrm{RCH}(X, 1/2)$



$\mathrm{RCH}(X, 0.3)$

Previous Results:

The existing literature looks at the asymptotic complexity of random convex hulls.

Theorem (Renyi and Sulanke, 1963)

Let X consist of N points drawn independently from the Normal distribution in the plane and let $\mathbb{E}[V_N]$ denote the expected number of vertices of the convex hull of X. Then for large N,

$$\mathbb{E}[V_N] \sim 2\sqrt{2\pi \ln N}.$$

Remark: We do not specify the mean or variance of the Normal distribution, as $\mathbb{E}[V_N]$ is translation and scale invariant.

Main Results:

Theorem

Let $\mathbb{E}[V_{N,1/k}]$ denote the expected number of vertices of the reduced convex hull of N points drawn from the standard Normal distribution in the plane, with $\mu = 1/k$. Then for both fixed k and for $k = \mathcal{O}(N)$,

$$\mathbb{E}[V_{N,1/k}] \sim 2\sqrt{2\pi \ln \binom{N}{k}}.$$

Other Properties: We've found estimates for when $\mu = 1/k$ for some integer k, but what about for other values of μ?

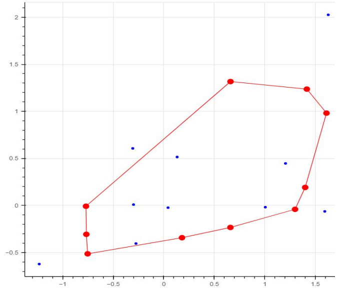
Proposition

Let $k = \lceil 1/\mu \rceil$. *Then* $V_{N,\mu} \leq (k+1) \cdot V_{N,1/k}$.

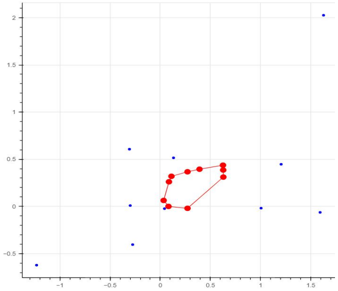
Since our asymptotic estimate depends on $\binom{N}{k}$, we may expect the symmetry between $\binom{N}{k}$ and $\binom{N}{N-k}$ implies a symmetry between $\mathrm{RCH}(X, 1/k)$ and $\mathrm{RCH}(X, 1/(N-k))$.

Proposition

The reduced convex hulls of N *points with parameters* μ *and* $\mu/(\mu N - 1)$ *are similar.* *RCH*(X, μ) *is RCH*$(X, \mu/(\mu N - 1))$ *rotated* 180° *about the centroid of* X *and scaled by a factor of* $1/(\mu N - 1)$.



$N = 10, \mu = 1/2$



$N = 10, \mu = 1/8$

Combining these propositions, we get a better picture of $V_{N,\mu}$ as a function of μ:

$V_N(1/\mu)$ is locally constant between k and $k+1$ for each integer $k = 1, 2, \ldots, N$, with point discontinuities at each k. The function is essentially non-decreasing from $1/\mu = 1$ to $1/\mu = N/2$, and is mirrored around the line $1/\mu = N/2$. Therefore the function reaches its maximum at $1/\mu \approx N/2$