Properties of Reduced Convex Hulls

Ben Arora
University of Connecticut

Abstract: We explore the properties of randomly-generated reduced convex hulls, both when the number of underlying points is fixed and in the asymptotic case. The results allow us to describe the number of vertices of a random reduced convex hull as either a function of the number of points or the hull parameter $\mu$.

Background Information: Given a set of points $X = \{x_1, x_2, \ldots, x_N\}$ we define the convex hull of $X$ to be the smallest convex set containing $X$, or equivalently,

$$CH(X) = \left\{ \sum_{i=1}^{N} \alpha_i x_i : \alpha_i \geq 0, \sum_{i=1}^{N} \alpha_i = 1 \right\}.$$
Sometimes we want to “shrink” the convex hull of a set of points in a non-uniform way. Given the same set of points $X$, we define the reduced convex hull of $X$ to be

$$\text{RCH}(X, \mu) = \left\{ \sum_{i=1}^{N} \alpha_i x_i : 0 \leq \alpha_i \leq \mu, \sum_{i=1}^{N} \alpha_i = 1 \right\}.$$  

Notice when $\mu = 1$ we obtain the full convex hull of $X$. Reduced convex hulls originated in the study of binary classification problems for which the data sets overlap.
**Previous Results:**
The existing literature looks at the asymptotic complexity of random convex hulls.

**Theorem (Renyi and Sulanke, 1963)**

Let $X$ consist of $N$ points drawn independently from the Normal distribution in the plane and let $E[V_N]$ denote the expected number of vertices of the convex hull of $X$. Then for large $N$,

$$E[V_N] \sim 2\sqrt{2\pi \ln N}.$$

**Remark:** We do not specify the mean or variance of the Normal distribution, as $E[V_N]$ is translation and scale invariant.

**Main Results:**

**Theorem**

Let $E[V_{N,1/k}]$ denote the expected number of vertices of the reduced convex hull of $N$ points drawn from the standard Normal distribution in the plane, with $\mu = 1/k$. Then for both fixed $k$ and for $k = \mathcal{O}(N)$,

$$E[V_{N,1/k}] \sim 2\sqrt{2\pi \ln \left(\frac{N}{k}\right)}.$$
**Other Properties:** We’ve found estimates for when $\mu = 1/k$ for some integer $k$, but what about for other values of $\mu$?

**Proposition**

Let $k = \lceil 1/\mu \rceil$. Then $V_{N, \mu} \leq (k + 1) \cdot V_{N, 1/k}$.

Since our asymptotic estimate depends on $\binom{N}{k}$, we may expect the symmetry between $\binom{N}{k}$ and $\binom{N}{N-k}$ implies a symmetry between $\text{RCH}(X, 1/k)$ and $\text{RCH}(X, 1/(N - k))$.

**Proposition**

*The reduced convex hulls of $N$ points with parameters $\mu$ and $\mu/(\mu N - 1)$ are similar. $\text{RCH}(X, \mu)$ is $\text{RCH}(X, \mu/(\mu N - 1))$ rotated $180^\circ$ about the centroid of $X$ and scaled by a factor of $1/(\mu N - 1)$.***

\[ N = 10, \mu = 1/2 \quad \text{and} \quad N = 10, \mu = 1/8 \]
Combining these propositions, we get a better picture of $V_{N,\mu}$ as a function of $\mu$:

$V_N(1/\mu)$ is locally constant between $k$ and $k + 1$ for each integer $k = 1, 2, \ldots, N$, with point discontinuities at each $k$. The function is essentially non-decreasing from $1/\mu = 1$ to $1/\mu = N/2$, and is mirrored around the line $1/\mu = N/2$. Therefore the function reaches its maximum at $1/\mu \approx N/2$. 

![Graph showing the behavior of $V_{18}(1/\mu)$](image)