Definition

Let $w(x)$ be a continuous function, and let f and g be polynomials. We say f and g are *orthogonal* on the interval $[a, b]$ with respect to the weight function $w(x)$ whenever

$$\int_a^b f(x)g(x)\,w(x)dx = 0.$$

This integral can be understood as an inner product of functions:

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,w(x)dx.$$

Consider the usual basis $\{1, x, x^2, \dots\}$ for the space of polynomials with coefficients in $\mathbb{R}$. We can use the Gram-Schimdt process from linear algebra on this basis with this inner product to find an orthogonal basis for this space.

When we apply Gram-Schimdt in case $w(x) = 1$, we obtain one normalization of the Legendre polynomials. We list the first few:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = 3x^2 - 1, \quad P_3(x) = 5x^3 - 3x.$$

If $n \neq m$, then $\boxed{\langle P_n, P_m \rangle = \int_{-1}^{1} P_n(x) P_m(x)\, dx = 0}$.

The Legendre polynomials $P_n(x)$ satisfy the differential equation

$$(1 - x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0.$$

We can give an alternative definition of Legendre polynomials:

Definition

A Legendre polynomial of degree n is a polynomial solution of

$$(1 - x^2) y'' - 2xy' + n(n+1) y = 0.$$

Using the Gram-Schimdt construction guarantees orthogonality. How can we prove orthogonality using the differential equation?

Byerly's trick: Consider the differential equations for P_n and P_m. We multiply each by P_m and P_n, respectively, and then subtract them. Writing $C = n(n+1) - m(m+1)$, this gives us

$$C \cdot P_n P_m = 2x(P_n' P_m - P_m' P_n) - (1 - x^2)(P_n'' P_m - P_m'' P_n).$$

If $y = P_n' P_m - P_m' P_n$, then $y' = P_n'' P_m - P_m'' P_n$. So, we have

$$C \cdot P_n(x) P_m(x) = 2xy(x) - (1 - x^2) y'(x).$$

Now we can integrate over $[t, 1]$ to find $\int_t^1 P_n(x) P_m(x)\, dx$.

Integrating over $[t, 1]$ leaves us with

$$C \int_t^1 P_n(x) P_m(x)\, dx = \int_t^1 2x y(x)\, dx - \int_t^1 (1 - x^2) y'(x)\, dx.$$

Now we integrate the right-most integral by parts:

$$\int_t^1 (1 - x^2) y'(x)\, dx = -(1 - t^2) y(t) + \int_t^1 2x y(x)\, dx.$$

A cancellation occurs! We're left with

$$C \int_t^1 P_n(x) P_m(x)\, dx = (1 - t^2) y(t).$$

As $n \neq m$, we can divide by C. Upon setting $t = -1$, we get $\displaystyle\int_{-1}^1 P_n(x) P_m(x)\, dx = 0$, as we'd hoped to show.

Takeaway

A very similar approach can be applied to all the classical orthogonal polynomials to show they are orthogonal on $[-1, 1]$. Examples of such polynomials include:

1. Chebyshev polynomials
2. Ultraspherical and Jacobi polynomials
3. Hermite polynomials
4. Laguerre polynomials.

Google the differential equations for these polynomials and try for yourself! The approach is adapted from the treatment of Legendre polynomials shown on page 172 of

W. E. Byerly, *An elementary treatise on Fourier's series and spherical, cylindrical, and ellipsoidal harmonics, with applications to problems in mathematical physics*, Dover Publications, Inc., New York, 1959.