Agmon-type decay of eigenfunctions for Schrödinger operators with a non-compactly supported classical region

Hengrui Zhu
Based on joint work with Christoph Marx

Oberlin College, Department of Mathematics

Abstract: An important result by Agmon [1;2] implies that an eigenfunction of a Schrödinger operator on $\mathbb{R}^n$ decays exponentially if the classical region is compactly supported. We extend this result to Schrödinger operators with eigenvalues, for which the **classical region is not compactly supported**. We show that assuming integrability of the classical region with respect to an increasing weight function implies L^2-decay of the eigenfunction with respect to the same weight. Here, decay is measured in the Agmon metric which takes into account anisotropies of the potential.

Background: A Schrödinger operator on $\mathbb{R}^n$ is given by

$$H = -\Delta + V \ ,$$

where Δ is the Laplacian, and $V : \mathbb{R}^n \to \mathbb{R}$ is the potential. We assume V to be bounded below and continuous. For an eigenfunction ψ of H with eigenvalue E, the classical region is defined by the set where $E > V$. Physically, it represents the region that could be reached by a classical particle with energy E.

For a continuous potential V and eigenvalue E, the Agmon metric is defined by [1;3]:

$$\rho_E(x,y) := \inf_{\gamma \in P_{x,y}} \int_0^1 [V(\gamma(t)) - E]_+^{1/2} \, ||\dot{\gamma}(t)|| \mathrm{d}t \ ,$$

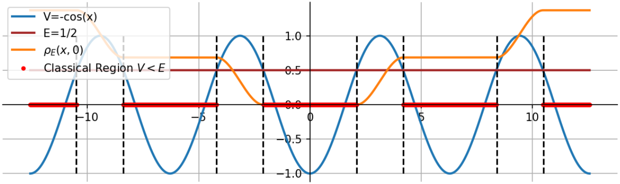
where $P_{x,y} := \{\gamma : [0,1] \to \mathbb{R}^n | \gamma(0) = x, \ \gamma(1) = y, \text{ and } \gamma = AC[0,1]\}$, and $[V - E]_+ := \max\{V - E, 0\}$. In particular, the distance in the Agmon metric is **0 precisely between two points connected by a path inside the classical region**.

In one dimension, the Agmon metric reduces to a simple form, known as the WKB factor:

$$\rho_E(x,y) := \int_x^y [V(t) - E]_+^{1/2} \, \mathrm{d}t \ .$$

Example

The Agmon distance from the origin, $\rho_E(x,0)$, is plotted for $V = -\cos(x)$ and $E = 1/2$. The bold segments in red on the x axis indicate the classical region.



Main Result: We show that, assuming **integrability of the classical region** with respect to a weight function ϕ, the **eigenfunctions decay minimally at the rate of ϕ in an $\mathcal{L}^2$ sense**. Further, if V is regular enough, this $\mathcal{L}^2$ bound implies a pointwise bound using the Sobolov Embedding theorem.

Hypotheses

(H1) Let $\psi \in D(V) \cap \mathcal{L}^\infty(\mathbb{R}^n)$, where $D(V) := \{f \in \mathcal{L}^2 : Vf \in \mathcal{L}^2\}$, be an eigenfunction of H such that $H\psi = E\psi$, for some $E \in \mathbb{R}$.
(H2) Given a weight function $1 \leq \phi \in C^2([0, +\infty))$ satisfying that

$$\lim_{x \to \infty} \phi(x) = \infty \ , \ \text{and that } 0 < \sup \left| \frac{\phi'}{\phi} \right| =: M_\phi < \infty \ , \tag{1}$$

there exist

$$\max \left\{ 0, 1 - M_\phi^{-2} \right\} < \epsilon < 1 \ \text{ and } \ \delta > 0 \ , \tag{2}$$

such that, for $\rho_E(x) := \rho_E(x, 0)$,

$$\chi_{\{V \leq E+\delta\}} \phi\left((1-\epsilon)\rho_E\right) \in \mathcal{L}^2(\mathbb{R}^n) \ . \tag{3}$$

Theorem (C. Marx and H. Zhu, 2020 [4])

Let $H = -\Delta + V$ be a closed operator with V bounded below and continuous, so that $\sigma(H) \subset \mathbb{R}$. Let ψ be as in (H1) with eigenvalue E satisfying (H2) for some weight function ϕ obeying (1). Then, for each ϵ as in (2), there exists a constant c_ϵ, $0 < c_\epsilon < \infty$, such that

$$\int \phi((1-\epsilon)\rho_E(x))^2 |\psi(x)|^2 \mathrm{d}x \leq c_\epsilon \ .$$

Sketch of Proof:

Step 1: We **break the integral in (4) into contributions from the classical and the non-classical region**:

$$\int \phi((1-\epsilon)\rho_E(x))^2|\psi(x)|^2\mathrm{d}x = \left\{\int_{V\geq E+\delta} + \int_{V<E+\delta}\right\}\phi((1-\epsilon)\rho_E(x))^2|\psi(x)|^2\mathrm{d}x\,, \tag{4}$$

The second integral is controlled by the integrability assumption in (H2).

Step 2: To bound the first integral, we **perform a gauge transform**, for $\alpha > 0$:

$$H \to H_{f_\alpha} := \phi(f_\alpha)H\phi(f_\alpha)^{-1}, \text{ where } f_\alpha := \frac{(1-\epsilon)\rho_E}{1+\alpha(1-\epsilon)\rho_E}\,, \tag{5}$$

and **prove a lower bound** for $\Re\langle\Phi_\alpha,(H_{f_\alpha}-E)\Phi_\alpha\rangle$, where $\Phi_\alpha := \phi(f_\alpha)\psi$:

$$||\Phi_\alpha||_2^2 \leq \frac{1}{\eta_\epsilon\delta}\left\{\Re\langle\Phi_\alpha,(H_{f_\alpha}-E)\Phi_\alpha\rangle + \int|\Phi_\alpha|^2(V-E)_-\right\} + \int_{\{V\leq E+\delta\}}|\Phi_\alpha|^2\,, \tag{6}$$

where $\eta_\epsilon := 1 - M_\phi^2(1-\epsilon)$ with M_ϕ as defined in (H2).

Step 3: We show that the right hand side of (6) is **uniformly bounded in** $\alpha > 0$. Hence, we can **take the limit** $\alpha \to 0^+$ and conclude, by Fatou's Lemma, that the first integral in (4) is also bounded.

Reference:

[1] S. Agmon: Lectures on exponential decay of solutions of second-order elip-tic equations. Princeton, NJ: Princeton University Press, 1982.

[2] P. D. Hislop: Exponential Decay of two-body eigenfunctions: A review. Electronic Journal of Differential Equations, Conf. 04, 2000, pp. 265–288

[3] P. D. Hislop, I. M. Sigal: Introduction to spectral theory, with applications to Schrödinger operators. New York: Springer, 1996.

[4] C. Marx, H. Zhu: Agmon-type decay of eigenfunctions for Schrödinger operators with a non-compactly supported classical region. In preparation, 2020.