Chaotic Behavior in Asymmetric Lemon Billiards

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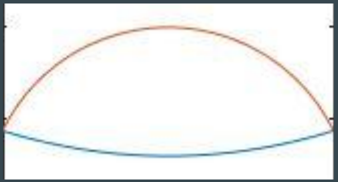
Abstract:
Chaotic dynamics emerge in mathematical billiards when the boundary of the billiard table is dispersing or when it contains focusing arcs at sufficient distance to allow a defocusing effect to occur. This project studies a type of billiard known as an asymmetric lemon billiard, comprised of focusing boundaries which seem to violate the usual defocusing condition.

Background:

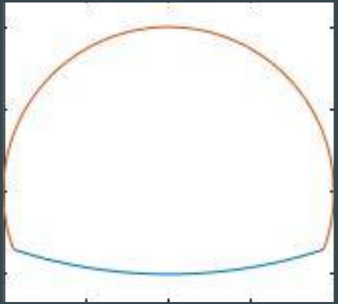
Mathematical billiards are important models of dynamical systems from mathematical physics in which point particles collide elastically with fixed boundaries. By studying these billiards we can describe the evolution of the system over long periods of time both numerically and graphically.

A hyperbolic system with a defocusing boundary has a concave boundary. After colliding with these boundaries, trajectories will defocus and diverge if they do not collide with another boundary too soon.

The asymmetric lemon billiards have a boundary made of two circular arcs of different radii with a shorter free flight time than in a regular circle; therefore, the usual criterion implies there is not enough space to defocus.



Surprisingly, we still observe defocusing for many choices of parameters describing the radii and distance between boundaries.



Parameters: We set the smaller radius $r=1$. We varied the larger radius $R > 1$ and the distance d between the centers of the two circles: $d-1 < R < d+1$ to make the arcs intersect.

Period 2 Orbits

Goal: to find the parameter values where the system transitions from unstable to stable

There are theoretical proofs that determine a billiard to be hyperbolic when the table contains a major arc of the circle of smaller radius. However, we still saw defocusing while the flight time was not double the radius of curvature.

We explored these billiard tables numerically to find sharper results.

As we increased the distance between the circular boundaries, we found that once the period two orbit went from stable to unstable, all other stable orbits were also broken.

Numerical Evidence

Once the period two orbit of the asymmetric lemon billiard became unstable, the rest of the table became hyperbolic. This is the case when $d < R$

The Lyapunov Exponent is the exponential rate of growth of expansion along an orbit. It is one measure of chaos. As the exponent increases, nearby trajectories diverge more rapidly.

We found the Lyapunov exponent changed as the distance between radi increased and decreased.

The exponent was calculated each time we increased the distance between radi, starting from distance Radius - 1 until Radius + 1, analyzing the whole range of circle crossover, producing the lemon.

Lyapunov Exponent

Larger radius = 3   smaller radius = 1   distance = 3.9 → 2.001